Pioneer Journal of Mathematics and Mathematical Sciences
Volume 1, Number 2, 2011, Pages 131-135
This paper is available online at http://www.pspchv.com/content_PJMMS.html

ON A CERTAIN INTEGRAL UNIVALENT OPERATOR

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Abstract

In this paper, author proves some properties of a certain integral operator.

1. Introduction

Let $H(U)$ be the class of all analytic functions in the unit disk $U = \{z \in C : |z| < 1\}$. Let $A = \left\{f \in H(U), \text{ with } f(z) = z + \sum_{n=2}^{\infty} a_n z^n, z \in U\right\}$ be the class of analytic functions in U and $S = \{f \in A : f \text{ is univalent in } U\}$. S^* is the class of starlike functions in the unit disk, defined by

$$S^* = \left\{f \in H(u) : f(0) = f'(0) - 1 = 0, \operatorname{Re}\frac{zf'(z)}{f(z)} > 0, z \in U\right\}.$$

A function $f \in S$ is said to be *starlike* of order α, $0 \leq \alpha < 1$ if $\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha$, $z \in U$ and it is denoted by $S^*(\alpha)$.

Received January 21, 2011
2010 Mathematics Subject Classification: 30C45.
Keywords and phrases: univalence, starlike and convex

The class of convex functions in U denoted by S^c is defined by

$$S^c = \left\{ f \in H(u) : f(0) = f'(0) - 1 = 0,\ \mathrm{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0,\ z \in U \right\}.$$

A function $f \in S$ is convex function of order α, $0 \le \alpha < 1$ and denoted by $S^c(\alpha)$ if f satisfies the inequality

$$\mathrm{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha, \quad z \in U.$$

A function is said to be *uniformly convex* if and only if

$$\mathrm{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} \ge \left|\frac{zf''(z)}{f'(z)}\right| \quad z \in U \tag{1}$$

and this is denoted by $f \in UCV$

In 1973, Kudryashov investigated the maximum value of M such the inequality

$$\left|\frac{zf''(z)}{f'(z)}\right| \le M \tag{2}$$

implies that f is univalent in U. He showed that if $M = 3.05\cdots$ and $\left|\frac{zf''(z)}{f'(z)}\right| \le 3.05\cdots$, where M is the solution of the equation $8[M(M-2)^3]^{1/2} - 3(3-M)^2 = 12$, then f is univalent in U.

Also, Miller and Mocanu [1] showed that if $\left|\frac{zf''(z)}{f'(z)}\right| \le 2.8329$, the function f is starlike in U.

Furthermore, Breaz et al. [1] consider the integral operator

$$F_{\alpha_1, \alpha_2, \ldots, \alpha_n}(z) = \int_0^z (f_1'(t))^{\alpha_1} \cdots (f_n'(t))^{\alpha_n}\, dt$$

and showed that

$$\left|\frac{zf_i''(z)}{f_i'(z)}\right| \le M_1 \tag{3}$$

starlike, where $M_1 = 2.8329\cdots$ is the smallest root of equation $x \sin x$ ...

2. Main Results

Theorem 2.1. *Let* $\alpha_k \in C$, $k \in \{1, 2, \ldots, n\}$ *and* $|\alpha_k| = (x_k^2 + y_k^2) > 0$ *consider* $f_k \in S$ *and let* $\left|\frac{zf_k''(z)}{f_k'(z)}\right| \le M$, $M = 3.05\cdots$ *for all* $z \in U$. *If* $\sum_{k=1}^n (x_k^2 + y_k^2)^1/2 \le 1$, *then the integral operator*

$$F_{x_1+iy_1, \ldots, x_n+iy_n} = \int_0^z \prod_{k=1}^n (f_k'(t))^{x_k+iy_k}\, dt$$

is univalent.

Proof. Let $F_{x_1+iy_1, \ldots, x_n+iy_n} = \int_0^z \prod_{k=1}^n (f_k'(t))^{x_k+iy_k}\, dt$. Let

$$H(z) = z\frac{F''_{x_1+iy_1, \ldots, x_n+iy_n}(z)}{F'_{x_1+iy_1, \ldots, x_n+iy_n}(z)} = z\left((x_1 + iy_1)\frac{f_1''(z)}{f_1'(z)} + \cdots + (x_n + iy_n)\frac{f_n''(z)}{f_n'(z)}\right),$$

$$|H(z)| \le z\left((x_1^2 + y_1^2)^{1/2}\left|\frac{f_1''(z)}{f_1'(z)}\right| + \cdots + (x_n^2 + y_n^2)^{1/2}\left|\frac{f_n''(z)}{f_n'(z)}\right|\right). \tag{4}$$

Applying inequality (2) to (4), we obtain $H(z) \le M\sum_{k=1}^n (x_k^2 + y_k^2)^1/2$ and by hypothesis, $H(z) \le M$. This shows that the integral operator $F_{x_1+iy_1, \ldots, x_n+iy_n} = \int_0^z \prod_{k=1}^z (f_k'(t))^{x_k+iy_k}\, dt$ is univalent. This concludes the proof of Theorem 2.1.

Theorem 2.2. *Let* $\alpha_k = x_k + iy_k$, $k \in \{1, 2, \ldots, n\}$ *and* $|\alpha_k| > 0$, *let* f_i *be univalent and suppose that* $\left|\frac{zf_k''(z)}{f_k'(z)}\right| \le M_1$, $M_1 = 2.8329\cdots$. *Then the integral operator*

$$F_{x_1+iy_1, \ldots, x_n+iy_n} = \int_0^z \prod_{k=1}^n (f_k'(t))^{x_k+iy_k}\, dt$$

Proof. We have

$$H(z) = z\frac{F''_{x_1+iy_1,\dots,x_n+iy_n}(z)}{F'_{x_1+iy_1,\dots,x_n+iy_n}(z)} = z\left((x_1+iy_1)\frac{f_1''(z)}{f_1'(z)} + \cdots + (x_n+iy_n)\frac{f_n''(z)}{f_n'(z)}\right).$$

Thus $H(z) \le M_1 \sum_{k=1}^{n} (x_k^2 + y_k^2)^1/2 \le M_1$. This implies that the integral operator

$F_{x_1+iy_1,\dots,x_n+iy_n} = \int_0^z \prod_{k=1}^{n} (f_k'(t))^{x_k+iy_k}\,dt$ is starlike.

Theorem 2.3. *Let* $\alpha_k = x_k + iy_k$, $k \in \{1, 2, \dots, n\}$ *and* $\operatorname{Re}\alpha_k = x_k > 0$. *Suppose* f_k *is convex for all* $k \in \{1, 2, \dots, n\}$. *Then the integral operator* $F_{x_1+iy_1,\dots,x_n+iy_n} = \int_0^z \prod_{k=1}^{n} (f_k'(t))^{x_k+iy_k}\,dt$ *is convex.*

Proof. Let f_k be convex. Then we have $\operatorname{Re}\left\{1 + z\frac{f_k''(z)}{f_k'(z)}\right\} > 0$. But

$$\operatorname{Re}\left\{1 + \frac{zF''_{x_1+iy_1,\dots,x_n+iy_n}(z)}{F'_{x_1+iy_1,\dots,x_n+iy_n}(z)}\right\} = \operatorname{Re}\left\{\sum_{k=1}^{n}\alpha_k\frac{zf_k''}{f_k'} + 1\right\}$$

$$= \sum_{k=1}^{n} x_k \operatorname{Re}\left(\frac{zf_k''}{f_k'} + 1\right) > 0,$$

for all $k \in \{1, 2, \dots, n\}$. Thus, the integral operator

$$F_{x_1+iy_1,\dots,x_n+iy_n} = \int_0^z \prod_{k=1}^{z} (f_k'(t))^{x_k+iy_k}\,dt$$

convex.

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