Far East Journal of Mathematical Sciences (FJMS)
Volume 46, Number 2, 2010, Pages 107-111
This paper is available online at http://pphmj.com/journals/fjms.htm

A CHARACTERIZATION OF ANALYTICITY USING CAUCHY INTEGRAL FORMULA

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Abstract

In this paper, we obtain a general characterization of analyticity of function of a complex variable using Cauchy integral formula.

1. Introduction

Important applications of Cauchy's theorem as well as that of its converse (Morera's theorem) have being noted by many authors. Among numerous applications of Cauchy's theorem is the Cauchy's integral formula. Ko et al. in [2] obtained a new characterization of the analytic functions by showing the converse of the Cauchy integral formula. In this paper, the author gives a generalization of the characterization of the analytic function using Cauchy integral formula.

We shall need the following results in the proof of our main results.

Cauchy integral formula [1-4]. Let f be analytic on a domain D and that γ is a positively oriented simple closed contour in D whose inside Ω also lies in D. Then

$$f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{w - z} dw, \quad \text{for all} \quad z \in \Omega.$$

2010 Mathematics Subject Classification: 30C45.

Keywords and phrases: analytic.

Received September 21, 2010

Lemma 1 [2]. *Let $f(z)$ be continuous, not necessary analytic on a contour γ, which is rectifiable in a domain D. Then the function*

$$F(z) = \int_\gamma \frac{f(w)}{w - z} dw$$

is analytic on D/γ, with

$$F'(z) = \int_\gamma \frac{f(w)}{(w - z)^2} dw.$$

Lemma 2 [2]. *Let $f(z)$ be continuous on a closed contour γ contained in a domain D. Then the following conditions are equivalent*:

(i) *$f(z)$ is analytic in D/γ,* (ii) *$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w - z} dw$ for each z not on γ in D not on C in D.*

2. Main Results

We shall now give the statements and proofs of our main results.

Theorem 1. *Let $f(z)$ be continuous but not necessary analytic on a contour which is rectifiable in a domain D. Then the function $F(z) = \int_\gamma \frac{f(w)}{w - z} dw$ is analytic on D/γ and its nth derivative exists with*

$$F^n(z) = n \int_\gamma \frac{f(w)}{(w - z)^{n+1}} dw.$$

Proof. Let z be an arbitrary point not on C in D. Let the minimum distance of z to C be d and let the length of contour be L and $|f(z)| \leq M,$ for all $z \in C.$ $F(z)$ is analytic by Lemma 1, it is left to show that

$$F^{(n)}(z) = n \int_\gamma \frac{f(w)}{(w - z)^{n+1}} dw.$$

Let the statement be true for $n = 1$ as it is shown in Lemma 1. Now, if the statement is true for $n = k,$ $k \geq 2.$ The truth of $k = 2$ is shown in [1, 3, 4] and in most of the complex analysis text. We shall now show for $n = k + 1,$ $k \geq 2$ for

$k = 2,$ we need to show that

$$F'''(z) = \int_\gamma \frac{f(w)}{(w - z)^4} dw.$$

Now, by the definition of differentiation, we have

$$F'''(z) = \lim_{s \to 0} \frac{F''(z + s) - F''(z)}{s}$$

$$= \frac{1}{s} \int_\gamma f(w) \frac{1}{(w - z - s)^3} - \frac{1}{(w - z)^3} dw$$

$$= \frac{1}{s} \int_\gamma f(w) \frac{(w - z)^3 - (w - z)^3 - 3(w - z)^2 s + 3(w - z)s^2 - s^3}{(w - z - s)^3 (w - z)^3} dw$$

$$\int_\gamma f(w) \frac{3(w - z)^2 - 3(w - z)s - s^2}{(w - z - s)^3 (w - z)^3}.$$

But

$$\frac{F''(z + s) - F''(z)}{s} - 3 \int_\gamma f(w) \frac{f(w)^4}{w - z} dw$$

$$= \int_\gamma f(w) \frac{3(w - z)^2 - 3(w - z)s - s^2}{(w - z - s)^3 (w - z)^3 - \left(\frac{3}{w - z}\right)^4} dw$$

$$= \int_\gamma f(w) \frac{3w - z^3 - 3w - z^2 s - w - zs^2 - 3w - z - s^3}{w - z - s^3 w - z^4} dw,$$

$$\left| \frac{F''(z + s) - F''(z)}{s} - 3 \int_\gamma \frac{f(w)^4}{w - z} dw \right| \leq \frac{2|s|M6d^2 - 9d|s| + 3|s|^2 L}{d^7}$$

Thus, as $s \to 0,$ we have $\left| \frac{F''(z + s) - F''(z)}{s} - 3 \int_\gamma \frac{f(w)^4}{w - z} dw \right| = 0.$ So by induction, we have $F^{(n)}(z) = n \int_\gamma \frac{f(w)^{n+1}}{w - z} dw.$ Thus Theorem 1 is proved.

Remarks. (i) The last inequality in the proof of Theorem 1 is as a result of the fact that, the minimum distance of $N(z)$ to the contour γ is $d/2$, that is, $|w - z - s| \geq d/2$ and $|w - z| \geq d$, for all $w \in \gamma$, (ii) Theorem 1 generalises Lemma 1 on page 108.

Theorem 2. *Let $f(z)$ be continuous on a closed contour γ contained in a domain D. Then the following conditions are equivalent:* (a) *$f(z)$ is analytic on D/γ,* (b) $f^{(n)}(z) = \frac{n!}{2\pi i}\int_\gamma \frac{f(w)}{w - z}{}^{n+1} dw$*, for all z not on γ in D*

Proof. Let $f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w - z} dw$. Then $f^{(n)}(z) = \frac{1}{2\pi i}\int_\gamma \frac{f^{(n)}(w)}{w - z} dw$.

Integrating $\frac{1}{2\pi i}\int_\gamma \frac{f^{(n)}(w)}{w - z} dw$ by parts n times (boundary terms vanish), we have

$$\frac{1}{2\pi i}\int_\gamma \frac{f^{(n)}(w)}{w - z} dw = \frac{1}{2\pi i}\left((w - z)^{-1} f^{(n-1)}(w) + 2\int_\gamma f^{k-2}(w)(w - z)^{-3} + \cdots \right.$$
$$\left. + n\int_\gamma f^{(n-(n-1))}(w)(w - z)^{-n}\right) dw$$

for $n = 3$, we obtain

$$\frac{1}{2\pi i}\int_\gamma \frac{f'''(w)}{w - z} dw = \frac{1}{2\pi i}(w - z)^{-1} f''(w) + (w - z)^{-2} f'(w) + 2((w - z)^{-3} f'(w))$$
$$+ 3\int_\gamma \frac{f(w)}{(w - z)^4} dw.$$

The last equality shows that

$$f'''(w) = \frac{}{2\pi i}\int_\gamma \frac{(w)}{(w - z)^4} dw \text{ (the boundary terms vanish),}$$

$$f'''(w) = \frac{3!}{2\pi i}\int_\gamma \frac{f(w)}{(w - z)^4} dw.$$

By induction, we have

$$f^{(n)}(w) = \frac{n!}{2\pi i}\int_\gamma \frac{f(w)}{(w - z)^{n+1}} dw.$$

This completes the proof of Theorem 2.

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