Hindawi Publishing Corporation
International Journal of Mathematics and Mathematical Sciences
Volume 2007, Article ID 12536, 13 pages
doi:10.1155/2007/12536

Research Article

Convergence of Solutions of Certain Fourth-Order Nonlinear Differential Equations

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Received 4 September 2006; Revised 14 December 2006; Accepted 29 March 2007

Recommended by Thomas P. Witelski

We give sufficient criteria for the existence of convergence of solutions for a certain class of fourth-order nonlinear differential equations using Lyapunov's second method. A complete Lyapunov function is employed in this work which makes the results to include and improve some existing results in literature.

1. Introduction

In this paper, we will consider the fourth-order differential equation

$$x^{(\text{iv})} + a\dddot{x} + f(x,\dot{x})\ddot{x} + g(\dot{x}) + h(x) = p(t), \tag{1.1}$$

where $a > 0$, the functions f, g, h, p are continuous in the respective arguments displayed explicitly, $\dot{x} = dx/dt$, $\ddot{x} = d^2x/dt^2$, $\dddot{x} = d^3x/dt^3$, and $x^{(\text{iv})} = d^4x/dt^4$. The conditions on f, g, h, and p are such that the existence of solutions of (1.1) corresponding to any preassigned initial solutions is guaranteed.

Solutions of the equation of the form (1.1) have been investigated by several researchers on the account of boundedness, stability, and global asymptotic stability (see, e.g., [1–9]). Some results on these can be found in [10]. Out of the numerous works on this class of equations only a few were devoted to the convergence of the solutions (see, e.g., [11, 12]).

By convergence of solutions we mean, given any two solutions $x_1(t)$ and $x_2(t)$ of (1.1), $x_2(t) - x_1(t) \to 0$, $\dot{x}_2(t) - \dot{x}_1(t) \to 0$, $\ddot{x}_2(t) - \ddot{x}_1(t) \to 0$, and $\dddot{x}_2(t) - \dddot{x}_1(t) \to 0$ as $t \to \infty$.

In [13–16], certain classes of third-order nonlinear differential equations were investigated and their solutions were proved to converge under certain conditions.

In [15], the author considered the equation

$$\dddot{x} + a\ddot{x} + b\dot{x} + h(x) = p(t,x,\dot{x},\ddot{x}) \tag{1.2}$$

and established that the boundedness of both $p(t)$ and $\int p(\tau)d\tau$ together with the differentiability of the function h guaranteed the convergence of the solutions of the considered equation. This result was improved upon in [16] when the stringent conditions placed on the function h in [15] were dispensed with.

Similarly in [14], the author established that the solutions of the considered equation converged without many restrictions on the nonlinear terms that were involved.

In [11], the author considered (1.1) with $g(\dot{x}) = c\dot{x}$ $(c > 0)$, and further with the assumption that h was not necessarily differentiable but satisfied an incrementary ratio $\eta^{-1}(h(x+\xi) - h(\xi))$ $\eta \neq 0$, which lies in a closed subinterval I_0 of the Routh-Hurwitz interval $(0,(ab-c)c/a^2)$, where $I_0 \equiv [\Delta_0, k(ab-c)c/a^2]$.

The author in [12] considered (1.1) with $f(x,\dot{x}) = b$ and criteria for the existence of convergent solutions were established, whereas in [11] he considered (1.1) with $f(x,\dot{x}) = b$ and $g(\dot{x}) = c$. The work in [12] extends [11] from equation with one nonlinearity to the one having two nonlinearities which makes it an extension of [11] as well as an extension of [15] to an analogous fourth-order equation.

In all these studies, Lyapunov's second method has been the main tool of investigation. In the literature, the incomplete Lyapunov functions are frequent and used by a quite appreciable number of researchers due to the nature of construction and simplicity. The works with the complete Lyapunov functions are not as frequent as the ones with incomplete Lyapunov function.

In this present work, we will extend the work in [14] to (1.1). With a suitable complete Lyapunov function and less stringent assumptions on the nonlinear terms f, g, h, and p, we will show that the solutions of (1.1) converge.

This work is organized in this order, the main result is presented in Section 2 as formulation of results. Section 3 deals with the tools needed to the proof of the main result. The proof of the main theorem is presented is Section 4.

2. Formulation of results

The following is the main result.

Theorem 2.1. *Suppose that $x_1(t)$ and $x_2(t)$ are two solutions of (1.1), suppose further that for arbitrary ξ, η $(\eta \neq 0)$,*

(i) $(h(\xi+\eta) - h(\xi))/\eta \in I_0$, $\eta \neq 0$;

(ii) $(g(\xi+\eta) - g(\xi))/\eta \neq 0$;

(iii) $h(0) = g(0) = 0$;

(iv) $|f(x,y)| \leq b$;

(v) $|p(t)| \leq \Lambda$, *(Λ constant)*

then there exists a positive constant K_5 such that

(vi) $S(t_2) \leq S(t_1)e^{-K_5(t_2-t_1)}$ *for* $t_2 \geq t_1$, *where*

$$S(t) = \left\{[x_2(t) - x_1(t)]^2 + [\dot{x}_2(t) - \dot{x}_1(t)]^2 + [\ddot{x}_2(t) - \ddot{x}_1(t)]^2 + [\dddot{x}_2(t) - \dddot{x}_1(t)]^2\right\}. \tag{2.1}$$

Furthermore, all solutions of (1.1) converge.

We have the following corollaries as the consequences of Theorem 2.1 when $x_1(t) = 0$ and $t_1 = 0$.

Corollary 2.2. *Suppose that $p = 0$ in (1.1) and suppose further that the conditions of the theorem hold, then the trivial solution of (1.1) is exponentially stable in the large.*

Corollary 2.3. *Suppose also that the conditions of Corollary 2.2 hold for arbitrary η ($\eta \neq 0$) and $\xi = 0$, then there exists a constant K_0 such that every solution $x(t)$ of (1.1) satisfies*

$$|x(t)| \le K_0, \qquad |\dot{x}(t)| \le K_0, \qquad |\ddot{x}(t)| \le K_0, \qquad |\dddot{x}_2(t)| \le K_0. \tag{2.2}$$

Remark 2.4. The corresponding linear equation to (1.1) given as

$$x^{(iv)} + a\dddot{x} + b\ddot{x} + c\dot{x} + dx = p(t), \tag{$*$}$$

$d > 0$ and constants b, c (with $h(x) = dx$, $f(x,\dot{x}) = b$, $g(\dot{x}) = c\dot{x}$) and $p(t) = 0$ in (1.1), is known to have convergent solutions if the Routh-Hurwitz conditions/criteria $ab - c > 0$, $(ab - c)c - a^2d > 0$ hold.

Notations 2.5. Throughout this paper, K_3, K_4, and K_5 will denote finite positive constants whose magnitudes depend only on the constants a, b, c, d, δ, and Δ but are independent of solutions of (1.1). K_i's are not necessarily the same for each time they occur, but each K_i, $i = 1,2,\dots,5$ retains its identity throughout.

3. Preliminary results

On setting $\dot{x} = y$, $\dot{y} = z$, $\dot{z} = w$, (1.1) can be replaced by an equivalent system

$$\begin{aligned} \dot{x} = y, \qquad \dot{y} &= z, \qquad \dot{z} = w, \\ \dot{w} = -aw - f(x,y)&z - g(y) - h(x) + p(t). \end{aligned} \tag{3.1}$$

Following Cartwright [17] and Reissig et al. [10], a possible Lyapunov function is a quadratic function in the variables for which the coefficients are suitably chosen. In this regard, we will assume a Lyapunov function of the form

$$\begin{aligned} 2V(x,y,z,w) = {} & Ax^2 + By^2 + Cz^2 + Dw^2 + 2Exy + 2Fxz \\ & + 2Ixw + 2Jyz + 2Myw + 2Nzw. \end{aligned} \tag{3.2}$$

Our investigation rests mainly on the properties of the function

$$W(t) \equiv V(x_2(t) - x_1(t), y_2(t) - y_1(t), z_2(t) - z_1(t), w_2(t) - w_1(t)) \tag{3.3}$$

with $V(x(t), y(t), z(t), w(t))$ written as $V(x, y, z, w)$, where

$$
\begin{aligned}
A &= \frac{a\delta}{\Delta}\{(b+d)(c^2+d^2)[d(1-ad)-c]+d^3[a(b^2+d^2)+L]\},\\
B &= \frac{\delta}{\Delta}\{dL(abd+c)+a(b^2+d^2)[b(d-c)+cd]\\
&\qquad +[d(1-ad)-c][ad(b^2+c^2)-cd^2(b+1)+a^2bc]\},\\
C &= \frac{\delta}{\Delta}\{a(b^2+d^2)[d(1-ad+a^2c+d)-c]\\
&\qquad +d[c(a^2+b^2)-ab][d(1-ad)-c]+dL(a^2c+d)\},\\
D &= \frac{cd\delta}{\Delta}\{L+ab^2+(d-c)+ab[(1-ad)-c]\},\\
E &= \frac{ac\delta}{\Delta}\{d^2L+(b^2+d^2)(d-c)\},\\
F &= \frac{cd\delta}{bd\Delta}\{d^2L+ad^2(b^2+d^2)+[b(a^2+d^2)+d^2][ab^2d^2[d(1-ad)-c]]\},\\
I &= \frac{abc[d(1-ad-c)]\delta}{\Delta},\\
J &= \frac{abcd\delta}{\Delta}\{a(b^2+d^2)+L\},\\
M &= \frac{a\delta}{\Delta}\{d^2L+bd[d(1-ad)-c]+(b^2+d^2)(d-c)\},\\
N &= \frac{acd\delta}{\Delta}\{ab^2+d-c+L\},\\
\Delta &= abcd[d(1-ad)-c],\\
L &= b[ad+c[c(b+1)-c]],
\end{aligned}
\tag{3.4}
$$

with a, b, c, d positive and $[d(1-ad)-c]>0$ were obtained after solving the equations that arose when constructing the Lyapunov function.

Thus, W is equivalent to $V(x,y,z,w)$ with x, y, z, w replaced with x_2-x_1, y_2-y_1, z_2-z_1, and w_2-w_1, respectively.

Now, define W as

$$
\begin{aligned}
&2W(x_2-x_1,y_2-y_1,z_2-z_1,w_2-w_1)\\
&\quad = A(x_2-x_1)^2+B(y_2-y_1)^2+C(z_2-z_1)^2+D(w_2-w_1)^2\\
&\qquad +2E(x_2-x_1)(y_2-y_1)+2F(x_2-x_1)(z_2-z_1)\\
&\qquad +2I(x_2-x_1)(w_2-w_1)+2J(y_2-y_1)(z_2-z_1)\\
&\qquad +2M(y_2-y_1)(w_2-w_1)+2N(z_2-z_1)(w_2-w_1).
\end{aligned}
\tag{3.5}
$$

We will prove the following.

Lemma 3.1. *Suppose W is defined as in (3.5) and $W(0,0,0,0)=0$, then there exist constants K_1 and K_2 such that the inequalities*

$$\begin{aligned} &K_1\left((x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2+(w_1-w_2)^2\right)\\ &\qquad \le W \le K_2\left((x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2+(w_1-w_2)^2\right) \end{aligned} \tag{3.6}$$

hold.

Proof of Lemma 3.1. Clearly, $W(0,0,0,0)\equiv 0$.

By rearranging (3.5), we have

$$\begin{aligned} &2W(x_2-x_1,y_2-y_1,z_2-z_1,w_2-w_1)\\ &\quad=\left(\frac{\delta}{\Delta}\right)\bigg\{a[d(1-ad)]\bigg\{b[c(x_2-x_1)+d(y_2-y_1)+(w_2-w_1)]^2\\ &\qquad\qquad +d^2[(y_2-y_1)+b^3d^2(x_2-x_1)]^2\\ &\qquad\qquad +b^2d[(y_2-y_1)+a^2bd(x_2-x_1)]^2\\ &\qquad\qquad +acd\left[(z_2-z_1)+\frac{b^2d^3}{a}(x_2-x_1)\right]^2\bigg\}\\ &\qquad +dL\bigg\{[(z_2-z_1)+ac(x_2-x_1)]^2\\ &\qquad\qquad +ac^2\left[(z_2-z_1)+\frac{1}{a}(w_2-w_1)\right]^2\\ &\qquad\qquad +c\left[(y_2-y_1)+\frac{ad}{c}(w_2-w_1)\right]^2\\ &\qquad\qquad +ad^2\left[(x_2-x_1)+\frac{c}{d}(y_2-y_1)\right]^2\\ &\qquad\qquad +abd\left[(y_2-y_1)+\frac{c}{d}(z_2-z_1)\right]^2\bigg\}\\ &\qquad +ad(b^2+d^2)\bigg\{ad^2\left[(x_2-x_1)+\frac{c(d-c)}{ad^3}(y_2-y_1)\right]^2\\ &\qquad\qquad +a^2c\left[(z_2-z_1)+\frac{d}{a}(x_2-x_1)\right]^2\\ &\qquad\qquad +\frac{c}{a(b^2+d^2)}[(w_2-w_1)+a(z_2-z_1)]^2\\ &\qquad\qquad +b(d-c)\left[(y_2-y_1)+\frac{(w_2-w_1)}{b}\right]^2\\ &\qquad\qquad +c[(y_2-y_1)+ab(z_2-z_1)]^2\bigg\}\\ &\qquad +\bigg\{[d(1-ad)-c](ad(c^2+d^2)+abd^2)-\frac{cd^3}{a}(b^2+d^2)\\ &\qquad\qquad -b^4cd^3-a^5b^4d^3-ab^6d^4-a^2c^2d^2L\bigg\}(x_2-x_1)^2 \end{aligned}$$

$$
\begin{aligned}
&+\Big\{[d(1-ad)-c][ad(b^2+c^2)-cd^2(b+1)+a^2bc-abd^2] \\
&\quad -ac^2dL-\frac{c^2(d-c)^2}{d^3}\Big\}(y_2-y_1)^2 \\
&+\{ad^2(b^2+d^2)+d(b^2c-ab)[d(1-ad)-c]-a^3b^2cd(b^2+d^2) \\
&\quad -abc^2L-a^2cd[ab^2+(d-c)]\}(z_2-z_1)^2 \\
&+\Big\{L-ab[d(1-ad)-c]-\frac{a}{b}(b^2+d^2)(d-c)-\frac{a^2d^3}{c}-cdL\Big\}(w_2-w_1)^2\Big\},
\end{aligned}
\tag{3.7}
$$

from which we obtain

$$
\begin{aligned}
&2W(x_2-x_1,y_2-y_1,z_2-z_1,w_2-w_1) \\
&\geq\left(\frac{\delta}{\Delta}\right)\Big\{\Big\{[d(1-ad)-c](ad(c^2+d^2)+abd^2)-\frac{cd^3}{a}(b^2+d^2) \\
&\qquad -b^4cd^3-a^5b^4d^3-ab^6d^4-a^2c^2d^2L\Big\}(x_2-x_1)^2 \\
&\qquad +\Big\{[d(1-ad)-c][ad(b^2+c^2)-cd^2(b+1)+a^2bc-abd^2]-ac^2dL \\
&\qquad -\frac{c^2(d-c)^2}{d^3}\Big\}(y_2-y_1)^2 \\
&\qquad +\Big\{ad^2(b^2+d^2)+d(b^2c-ab)[d(1-ad)-c] \\
&\qquad -a^3b^2cd(b^2+d^2)-abc^2L-a^2cd[ab^2+(d-c)]\Big\}(z_2-z_1)^2 \\
&\qquad +\Big\{L-ab[d(1-ad)-c] \\
&\qquad -\frac{a}{b}(b^2+d^2)(d-c)-\frac{a^2d^3}{c}-cdL\Big\}(w_2-w_1)^2\Big\} \\
&\geq K_1\left((x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2+(w_2-w_1)^2\right),
\end{aligned}
\tag{3.8}
$$

where

$$
\begin{aligned}
K_1=\frac{\delta}{\Delta}\min\Big\{&\Big|[d(1-ad)-c](ad(c^2+d^2)+abd^2)-\frac{cd^3}{a}(b^2+d^2) \\
&\quad -b^4cd^3-a^5b^4d^3-ab^6d^4-a^2c^2d^2L\Big|, \\
&\Big|[d(1-ad)-c][ad(b^2+c^2)-cd^2(b+1)+a^2bc-abd^2] \\
&\quad -ac^2dL-\frac{c^2(d-c)^2}{d^3}\Big|,
\end{aligned}
$$

$$\left| ad^2(b^2+d^2) + d(b^2c - ab)[d(1-ad) - c] - a^3b^2cd(b^2+d^2) - abc^2L - a^2cd[ab^2 + (d-c)] \right|, \left| L - ab[d(1-ad) - c] - \frac{a}{b}(b^2+d^2)(d-c) - \frac{a^2d^3}{c} - cdL \right| \Big\}. \tag{3.9}$$

Therefore,

$$\begin{aligned} &2W(x_2 - x_1, y_2 - y_1, z_2 - z_1, w_2 - w_1) \\ &\quad \geq K_1\left((x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 + (w_2 - w_1)^2\right). \end{aligned} \tag{3.10}$$

By using the the Schwartz inequality $|xy| \leq (1/2)\|x^2 + y^2\|$ on (3.2), we have

$$\begin{aligned} 2W&(x_2 - x_1, y_2 - y_1, z_2 - z_1, w_2 - w_1) \\ &\leq \left(\frac{\delta}{\Delta}\right)\Big\{[A+E+F+I](x_2 - x_1)^2 + [B+E+J+M](y_2 - y_1)^2 \\ &\qquad + [C+F+J+N](z_2 - z_1)^2 + [D+I+M+N](w_2 - w_1)^2\Big\} \\ &\leq K_2\left((x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 + (w_2 - w_1)^2\right), \end{aligned} \tag{3.11}$$

where

$$K_2 = \left(\frac{\delta}{\Delta}\right)\max\{[A+E+F+I], [B+E+J+M], [C+F+J+N], [D+I+M+N]\} > 0. \tag{3.12}$$

From inequalities (3.10) and (3.11), we have

$$\begin{aligned} &K_1\left((x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 + (w_2 - w_1)^2\right) \\ &\quad \leq W \leq K_2\left((x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 + (w_2 - w_1)^2\right). \end{aligned} \tag{3.13}$$

This proves Lemma 3.1.

Lemma 3.2. *Suppose that $(x_1(t), y_1(t), z_1(t), w_1(t))$, and $(x_2(t), y_2(t), z_2(t), w_2(t))$ are any two distinct solutions of system (3.1) such that*

$$H(x_1, x_2) = \frac{h(x_1(t)) - h(x_2(t))}{x_1(t) - x_2(t)} \in I_0, \qquad G(y_1, y_2) = \frac{g(y_1(t)) - g(y_2(t))}{y_1(t) - y_2}(t) \neq 0 \tag{3.14}$$

for all $t > 0$ $(0 < t < \infty)$, where I_0 carries its usual meaning as $I_0 = [\delta, \Delta]$, then the function

$$W = V(x_1 - x_2, y_1 - y_2, z_1 - z_2, w_1 - w_2) \tag{3.15}$$

satisfies

$$\dot{W} \le -K_3 W \tag{3.16}$$

for some $K_3 > 0$.

Proof of Lemma 3.2. Differentiating W with respect to t using system (3.1), we obtain after some simplifications

$$\begin{aligned}\dot{W} = \left(\frac{\delta}{\Delta}\right)\Big\{ & -Ih(x_1(t)-x_2(t))(x_1-x_2) - Mg(y_1(t)-y_2(t))(y_1-y_2) \\ & -[Nb-J](z_1-z_2)^2 - [Da-N](w_1-w_2)^2 - Ig(y_1(t)-y_2(t))(x_1-x_2) \\ & -Mh(x_1(t)-x_2(t))(y_1-y_2) - [Ib-E](x_1-x_2)(z_1-z_2) \\ & -Nh(x_1(t)-x_2(t))(z_1-z_2) - [Ia-F](x_1-x_2)(w_1-w_2) \\ & -Dh(x_1(t)-x_2(t))(w_1-w_2) - [Mb-F-B](y_1-y_2)(z_1-z_2) \\ & -Ng(y_1(t)-y_2(t))(z_1-z_2) - [Ma-I-J](y_1-y_2)(w_1-w_2) \\ & -Dg(y_1(t)-y_2(t))(w_1-w_2) - [Db+Na-M-C](z_1-z_2)(w_1-w_2) \\ & +E(y_1-y_2)^2 + A(x_1-x_2)(y_1-y_2) + p(t)[I(x_1-x_2)+M(y_1-y_2) \\ & +N(z_1-z_2)+D(w_1-w_2)]\Big\}.\end{aligned} \tag{3.17}$$

Using the conditions on $h(x_1-x_2)$ and $g(y_1-y_2)$, (3.17) becomes

$$\begin{aligned}\dot{W} \le \left(\frac{\delta}{\Delta}\right)\Big\{ & -IH(x_1,x_2)(x_1-x_2)^2 - MG(y_1,y_2)(y_1-y_2)^2 - [Nb-J](z_1-z_2)^2 \\ & -[Da-N](w_1-w_2)^2 - IG(y_1,y_2)(x_1-x_2)(y_1-y_2) \\ & -MH(x_1,x_2)(x_1-x_2)(y_1-y_2) - [Ib-E](x_1-x_2)(z_1-z_2) \\ & -NH(x_1,x_2)(x_1-x_2)(z_1-z_2) - NG(y_1,y_2)(y_1-y_2)(z_1-z_2) \\ & -[Mb-F-B](y_1-y_2)(z_1-z_2) - [Ia-F](x_1-x_2)(w_1-w_2) \\ & -DH(x_1,x_2)(x_1-x_2)(w_1-w_2) - [Ma-I-J](y_1-y_2)(w_1-w_2) \\ & -DG(y_1,y_2)(y_1-y_2)(w_1-w_2) - [Db+Na-M-C](z_1-z_2)(w_1-w_2) \\ & +E(y_1-y_2)^2 + A(x_1-x_2)(y_1-y_2) + p(t)[I(x_1-x_2)+M(y_1-y_2) \\ & +N(z_1-z_2)+D(w_1-w_2)]\Big\}.\end{aligned} \tag{3.18}$$

This can be written as

$$\dot{W} \leq -\frac{\delta}{\Delta} W, \tag{3.19}$$

where

$$W = \{W_1 + W_2 + W_3 + W_4 + W_5 + W_6 + W_7 + W_8 + W_9 + W_{10} + W_{11} + W_{12} - W_{13}\}, \tag{3.20}$$

with

$$\begin{aligned}
W_1 &= \alpha_1 H(x_1,x_2)(x_1 - x_2)^2 + \beta_1 MG(y_1,y_2)(y_1 - y_2)^2 + \gamma_1 (z_1 - z_2)^2 \\
&\quad + \eta_1 (w_1 - w_2)^2, \\
W_2 &= \alpha_2 H(x_1,x_2)(x_1 - x_2)^2 + IG(y_1,y_2)(x_1 - x_2)(y_1 - y_2) \\
&\quad + \beta_2 MG(y_1,y_2)(y_1 - y_2)^2, \\
W_3 &= \alpha_3 H(x_1,x_2)(x_1 - x_2)^2 + MH(x_1,x_2)(x_1 - x_2)(y_1 - y_2) \\
&\quad + \beta_3 MG(y_1,y_2)(y_1 - y_2)^2, \\
W_4 &= \alpha_4 H(x_1,x_2)(x_1 - x_2)^2 + [Ib - E](x_1 - x_2)(z_1 - z_2) + \gamma_2 (z_1 - z_2)^2, \\
W_5 &= \alpha_5 H(x_1,x_2)(x_1 - x_2)^2 + NH(x_1,x_2)(x_1 - x_2)(z_1 - z_2) + \gamma_3 (z_1 - z_2)^2, \\
W_6 &= \alpha_6 H(x_1,x_2)(x_1 - x_2)^2 + [Ia - F](x_1 - x_2)(w_1 - w_2) + \eta_2 (w_1 - w_2)^2, \\
W_7 &= \alpha_7 H(x_1,x_2)(x_1 - x_2)^2 + DH(x_1,x_2)(x_1 - x_2)(w_1 - w_2) + \eta_3 (w_1 - w_2)^2, \\
W_8 &= \beta_4 MG(y_1,y_2)(y_1 - y_2)^2 + [Mb - F - B](y_1 - y_2)(z_1 - z_2) + \gamma_4 (z_1 - z_2)^2, \\
W_9 &= \beta_5 MG(y_1,y_2)(y_1 - y_2)^2 + NG(y_1,y_2)(y_1 - y_2)(z_1 - z_2) \\
&\quad + \gamma_5 MG(y_1,y_2)(y_1 - y_2)^2, \\
W_{10} &= \beta_6 MG(y_1,y_2)(y_1 - y_2)^2 + [Ma - I - J](y_1 - y_2)(w_1 - w_2) \\
&\quad + \eta_4 (w_1 - w_2)^2, \\
W_{11} &= \beta_7 MG(y_1,y_2)(y_1 - y_2)^2 + DG(y_1,y_2)(y_1 - y_2)(w_1 - w_2) + \eta_5 (w_1 - w_2)^2, \\
W_{12} &= \gamma_6 (z_1 - z_2)^2 + [Db + Na - M - C](z_1 - z_2)(w_1 - w_2) + \eta_6 (w_1 - w_2)^2, \\
W_{13} &= [I(x_1 - x_2) + M(y_1 - y_2) + N(z_1 - z_2) + D(w_1 - w_2)]\,p(t), \\
&\sum_{i=1}^{7} \alpha_i = 1, \qquad \sum_{i=1}^{7} \beta_i = 1, \qquad \sum_{i=1}^{6} \gamma_i = 1, \qquad \sum_{i=1}^{6} \eta_i = 1,
\end{aligned} \tag{3.21}$$

$W_2, W_3, \dots, W_{12}$ are quadratic forms in the variables involved. For any quadratic form $AX^2 + BX + C$ to be positive, $B^2 \leq 4AC$. With this property, W_i's, $i = 2, 3, \dots, 12$, are positive if

$$\max\left\{\frac{(Ib-E)^2}{\alpha_4\gamma_2}, \frac{(Ia-F)^2}{4\alpha_6\eta_2}\right\} \leq H \leq \min\left\{\frac{4\alpha_5\gamma_3}{N^2}, \frac{4\alpha_7\eta_3}{D^2}\right\}, \tag{a}$$

$$\max\left\{\frac{(Mb-F-B)^2}{M\beta_4\gamma_4}, \frac{(Ma-I-J)^2}{4M\beta_6\gamma_4}\right\} \leq G \leq \min\left\{\frac{4M\beta_5\gamma_5}{N^2}, \frac{4M\beta_7\eta_5}{D^2}\right\} \tag{b}$$

(see the appendix for details).

Moreover, with suitable choice of δ (small enough), we can always have

$$W_{13} \geq \delta\left\{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2 + (w_2-w_1)^2\right\}^{1/2}. \tag{3.22}$$

With these conditions, we have that

$$\begin{gathered} W \geq W_1, \\ W_1 \leq K_3\left\{\left((x_1-x_2)^2 + (y_1-y_2)^2 + (z_1-z_2)^2 + (w_1-w_2)^2\right)\right\}, \end{gathered} \tag{3.23}$$

with $K_3 = \max\{\alpha_1 H(x_1,x_2), \beta_1 MG(y_1,y_2), \gamma_1, \eta_1\}$.

Then from (3.19), we could have a K_4 such that

$$\dot{W} \leq \left(\frac{\delta}{\Delta}\right)\left\{-K_4\left((x_1-x_2)^2 + (y_1-y_2)^2 + (z_1-z_2)^2 + (w_1-w_2)^2\right)\right\} \tag{3.24}$$

or

$$\dot{W} \leq -K_5 W, \tag{3.25}$$

with $K_5 = \delta/\Delta K_4$.

This completes the proof of Lemma 3.2. □

Since $x_1(t)$ and $x_2(t)$ are solutions to be considered, we want to establish that the two solutions converge. Next is to establish that the solutions $x_1(t)$ and $x_2(t)$ converge.

4. Proof of the main result

We will now give the proof of the main result.

Proof of Theorem 2.1. Indeed from inequality (3.25),

$$\frac{dW}{dt} \leq -K_5 W. \tag{4.1}$$

On integration from t_1 to t_2, we have that

$$\begin{gathered} \ln\left(\frac{W(t_2)}{W(t_1)}\right) \leq -K_5(t_2-t_1), \\ \frac{W(t_2)}{W(t_1)} \leq \exp-\left(K_5(t_2-t_1)\right). \end{gathered} \tag{4.2}$$

Therefore,

$$W(t_2) \leq W(t_1)\exp\left(K_5(t_2 - t_1)\right). \tag{4.3}$$

From inequality (3.23), it follows that

$$W_1 \leq K_3 S, \tag{4.4}$$

where S is as defined in Theorem 2.1. From Lemma 3.1, we have that

$$\begin{aligned} W(t1) &\leq K_2\left((x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2 + (w_1 - w_2)^2\right) = K_2 S(t_1), \\ W(t2) &\leq K_2\left((x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2 + (w_1 - w_2)^2\right) = K_2 S(t_2); \end{aligned} \tag{4.5}$$

using this in inequality (4.3), we have

$$S(t_2) \leq S(t_1)\exp\left(-K_5(t_2 - t_1)\right) \tag{4.6}$$

for $t_2 \geq t_1$.

As $t \to \infty$,we have from inequality (4.3) that

$$\dot{W} \leq 0. \tag{4.7}$$

Also from inequality (4.6),

$$S(t_2) \longrightarrow 0 \quad \text{as } t_2 \longrightarrow \infty. \tag{4.8}$$

This implies that

$$\begin{aligned} x_2(t) - x_1(t) &\longrightarrow 0, \qquad \dot{x}_2(t) - \dot{x}_1(t) \longrightarrow 0, \\ \ddot{x}_2(t) - \ddot{x}_1(t) &\longrightarrow 0, \qquad \dddot{x}_1(t) - \dddot{x}_2(t) \longrightarrow 0. \end{aligned} \tag{4.9}$$

Hence the proof of Theorem 2.1 is complete. □

Appendix

The W_i's, $i = 2,3,\ldots,12$, are positive if

$$\frac{G(y_1,y_2)}{H(x_1,x_2)} \leq \frac{4M\alpha_2\beta_2}{I^2}, \tag{A.1}$$

$$\frac{H(x_1,x_2)}{G(y_1,y_2)} \leq \frac{4\alpha_3\beta_3}{M}, \tag{A.2}$$

$$\frac{(Ib - E)^2}{\alpha_4\gamma_2} \leq H(x_1,x_2), \tag{A.3}$$

$$H(x_1,x_2) \leq \frac{4\alpha_5\gamma_3}{N^2}, \tag{A.4}$$

$$\frac{(Ia - F)^2}{4\alpha_6\eta_2} \leq H(x_1,x_2), \tag{A.5}$$

$$H(x_1,x_2) \leq \frac{4\alpha_7\eta_3}{D^2}, \tag{A.6}$$

$$\frac{(Mb-F-B)^2}{M\beta_4\gamma_4} \leq G(y_1,y_2), \tag{A.7}$$

$$G(y_1,y_2) \leq \frac{4M\beta_5\gamma_5}{N^2}, \tag{A.8}$$

$$\frac{(Ma-I-J)^2}{4M\beta_6\eta_4} \leq G(y_1,y_2), \tag{A.9}$$

$$G(y_1,y_2) \leq \frac{4M\beta_7\eta_5}{D^2}, \tag{A.10}$$

$$(Db+Na-M-C)^2 \leq 4\gamma_6\eta_6, \tag{A.11}$$

respectively.

Acknowledgment

The authors wish to express their profound gratitude to the anonymous referees for their comments, contributions, and suggestions that made this work reach this stage.

References

[1] J. O. C. Ezeilo, "A stability result for solutions of a certain fourth order differential equation," *Journal of the London Mathematical Society*, vol. 37, no. 1, pp. 28–32, 1962.

[2] J. O. C. Ezeilo, "New properties of the equation $x''' + a'' + bx' + h(x) = p(t;x,x',x'')$ for certain special values of the incrementary ratio $y^{-1}\{h(x+y)-h(x)\}$," in *Équations différentielles et fonctionnelles non linéaires (Actes Conférence Internat. "Equa-Diff73", Brussels/Louvain-la-Neuve, 1973)*, pp. 447–462, Hermann, Paris, France, 1973.

[3] M. Harrow, "A stability result for solutions of certain fourth order homogeneous differential equations," *Journal of the London Mathematical Society*, vol. 42, no. 1, pp. 51–56, 1967.

[4] B. S. Ogundare, "Boundedness of solutions to fourth order differential equations with oscillatory restoring and forcing terms," *Electronic Journal of Differential Equations*, vol. 2006, no. 6, pp. 1–6, 2006.

[5] C. Tunç, "A note on the stability and boundedness results of solutions of certain fourth order differential equations," *Applied Mathematics and Computation*, vol. 155, no. 3, pp. 837–843, 2004.

[6] C. Tunç, "Some stability and boundedness results for the solutions of certain fourth order differential equations," *Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica*, vol. 44, pp. 161–171, 2005.

[7] C. Tunç, "An ultimate boundedness result for a certain system of fourth order nonlinear differential equations," *Differential Equations and Applications*, vol. 5, pp. 163–174, 2005.

[8] C. Tunç, "Stability and boundedness of solutions to certain fourth-order differential equations," *Electronic Journal of Differential Equations*, vol. 2006, no. 35, pp. 1–10, 2006.

[9] C. Tunç and A. Tiryaki, "On the boundedness and the stability results for the solution of certain fourth order differential equations via the intrinsic method," *Applied Mathematics and Mechanics*, vol. 17, no. 11, pp. 1039–1049, 1996.

[10] R. Reissig, G. Sansone, and R. Conti, *Non-Linear Differential Equations of Higher Order*, Noordhoff, Leyden, The Netherlands, 1974.
[11] A. U. Afuwape, "On the convergence of solutions of certain fourth order differential equations," *Analele ştiinţifice ale Universităţii "Al. I. Cuza" din Iaşi. Seria Nouă. Secţiunea I a Matematică*, vol. 27, no. 1, pp. 133–138, 1981.
[12] A. U. Afuwape, "Convergence of the solutions for the equation $x^{(iv)} + a\dddot{x} + b\ddot{x} + g(\dot{x}) + h(x) = p(t; x, \dot{x}, \ddot{x}, \dddot{x})$," *International Journal of Mathematics and Mathematical Sciences*, vol. 11, no. 4, pp. 727–733, 1988.
[13] A. U. Afuwape, "On the convergence of solutions of certain systems of nonlinear third-order differential equations," *Quarterly Journal of Pure and Applied Mathematics*, vol. 57, no. 4, pp. 255–271, 1983.
[14] B. S. Ogundare, "On the convergence of solutions of certain third order non-linear differential equations," *Mathematical Sciences Research Journal*, vol. 9, no. 11, pp. 304–312, 2005.
[15] H. O. Tejumola, "On the convergence of solutions of certain third-order differential equations," *Annali di Matematica Pura ed Applicata*, vol. 78, no. 1, pp. 377–386, 1968.
[16] H. O. Tejumola, "Convergence of solutions of certain ordinary third order differential equations," *Annali di Matematica Pura ed Applicata*, vol. 94, no. 1, pp. 247–256, 1972.
[17] M. L. Cartwright, "On the stability of solutions of certain differential equations of the fourth order," *The Quarterly Journal of Mechanics and Applied Mathematics*, vol. 9, no. 2, pp. 185–194, 1956.

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