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ON HULL-MINIMAL IDEALS IN THE SCHWARTZ ALGEBRAS OF SPHERICAL FUNCTIONS ON A SEMI-SIMPLE LIE GROUP

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Abstract

The hull-minimal ideals are discussed for the Harish-Chandra Schwartz algebra of a connected semi-simple Lie group G. In particular, we construct the basis elements for these ideals, and, when restricted to the Schwartz algebra of τ-spherical functions on G, we characterize the hull-minimal ideals in terms of the cusp forms.

1. Introduction

Let G be a connected semi-simple Lie group with a Lie algebra, $\mathfrak{g}$, and a maximal compact subgroup K. Denote the universal enveloping algebra of the complexification $\mathfrak{g}_{\mathbb{C}}$, of $\mathfrak{g}$ by $U(\mathfrak{g}_{\mathbb{C}})$ and its center by $\mathfrak{Z}$. Define the Harish-Chandra Schwartz algebra, $\mathscr{C}(G)$, of G as

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$$\mathscr{C}(G) := \{f \in C^{\infty}(G) : \sup\{|f(b;\, x;\, a)|\Xi(x)^{-1}(1 + \sigma(x))^{r}\} =: \mu_{a,b,r}(f) < \infty,$$
$$\text{for all } a,\, b, \in U(\mathfrak{g}_{\mathbb{C}}) \text{ and } r \geq 0\}$$

Here Ξ and σ are well known zonal spherical functions on G and $f \mapsto afb$ $=: f(b;\, \cdot;\, a)$ is the action of $U(\mathfrak{g}_{\mathbb{C}})$ realized as an algebra of invariant differential operators on functions on G. We define $\mathscr{C}(G \,//\, K)$ and $\mathscr{C}_{\tau}(G)$ analogously, for any two-sided representation $\tau = (\tau_1, \tau_2)$ of K. It is well-known that $\mathscr{C}(G)$ is a Fréchet algebra on G under the collection of seminorms $\{\mu_{a,b,r}\}$ which generalizes the classical Schwartz algebra on $\mathbb{R}^n$, and that $\mathscr{C}(G \,//\, K)$ and $\mathscr{C}_{\tau}(G)$ are closed subalgebras. In particular, $\mathscr{C}(G \,//\, K)$ is commutative.

We now consider an ideal J in an abstract complex associative algebra A and when exactly it is called *hull-minimal*. Let $\mathrm{Prim}(A)$ (called the *primitive ideal space*) denote the set of all primitive ideals of A. These are the ideals which are the kernels of some algebraically irreducible representations of A on a vector space. Let the *hull*, $h(J)$, of any ideal J of A be given as $\{I \in \mathrm{Prim}(A) : I \supseteq J\}$ and define the kernel, $\ker(C)$, of any subset C of $\mathrm{Prim}(A)$, as $\ker(C) = \bigcap_{J \in C} J$. We shall call any subset C of $\mathrm{Prim}(A)$ closed whenever $C = h(I)$ for some ideals I of A. The pair $(\mathrm{Prim}(A), \tau)$ is a topological space when τ is the well-known Jacobson topology on $\mathrm{Prim}(A)$. Hence, an ideal J of A is called *hull-minimal* whenever

(a) $h(J) = C$, C is a closed subset of $\mathrm{Prim}(A)$, and

(b) $J \subseteq I$ for every other ideal I of A whose hull is contained in C.

Property (a) expresses the 'hullity' of J while (b) contains its 'minimality'. There is a candidate for the position of J in (b) above. If C is any preassigned closed subset of $\mathrm{Prim}(A)$, then $\bigcap_{h(I)=C} I =: \mathscr{N}(C)$ is well-defined and is contained in every member of C. However, on property (a) the best we can say is that $C \subseteq h(\mathscr{N}(C))$. It is known that equality in the last statement holds if the algebra A is a regular semi-simple commutative Banach algebra ([5, p. 84]). Hence we seek a general form of regularity on our abstract complex associative algebra A. This has led to the introduction of the notion of hull-kernel regularity ([6, p. 78]). The problem now is to know the sufficient conditions on A in order to make it hull-kernel regular.

We study in the next section of this paper these conditions, when A is a Fréchet algebra and thereafter seek a concrete realization for the hull-minimal ideals in the interesting case of $A = \mathscr{C}(G)$ in terms of $\mathfrak{Z}$-finite K-finite functions in Section 3. And since only the principal series of representations of G coming from the minimal parabolic subgroup of G contributes to the theory of zonal spherical functions, the philosophy of cusp forms implies that $\mathscr{C}(G // K)$ does not admit *type-I hull-minimal ideals* in our present frame of work so that one can concentrate on $\mathscr{C}(G)$ and in particular on $\mathscr{C}_\tau(G)$. This is the content of Section 4.

2. Hull-minimal Ideals in a Fréchet Algebra

The starting point of the subject is the following lemma of Ludwig et al. [7, p. 173] giving sufficient conditions for the hull-minimal ideal of an abstract associative algebra to contain a preassigned member of A. This lemma would be seen to lead directly to the notion of hull-kernel regular algebra.

Lemma 2.1 ([6, p. 78] and [7, p. 173]). *Let C be a closed subset of* $\mathrm{Prim}(A)$ *and suppose that there exist elements $a, b \in A$ such that $b \in \ker(C)$ and $b \cdot a = a$. Then every ideal I of A with $h(I) \subseteq C$ contains a.*

It is interesting to note that the ideal I in Lemma 2.1 above already satisfies the reverse inclusion, $h(I) \subseteq C$, that was lacking in the motivational case of $\mathscr{N}(C)$ in which we only have $C \subseteq h(\mathscr{N}(C))$ as seen above. Hence we now have the requirement (a) above and in order to always have condition (b) as well we strengthen the hypotheses of Lemma 2.1 by carving it into the following definition.

Definition 2.2. A semi-simple algebra A is said to be *hull-kernel regular* if for every closed subset C of $\mathrm{Prim}(A)$ and for every $J \in \mathrm{Prim}(A)\backslash C$, there exist $a = a_J$, $b = b_J \in A$ such that (i) $b \in \ker(C)$, and (ii) $b \cdot a = a$.

It is clear that every hull-kernel regular commutative Banach algebra is automatically regular, if we use the map $f_b : A \to A$ given as $f_b(a) = b \cdot a$, with a and b as above. In Theorem 2.3 and Propositions 2.6 and 2.9 below, we rework some results of Ludwig ([6, pp. 79 and 84]) which are important for what follows in Sections 3 and 4.

Theorem 2.3. *If A is any hull-kernel regular algebra and C is any closed subset* *of* Prim(A), *then the hull-minimal ideal* $j(C)$, *associated with* C, *exists, and is generated by the elements,* a_J, $J \notin C$ *(as in Definition* 2.2*).*

Proof. Let I be the ideal generated by the elements a_J, $J \notin C$. We claim that $h(I) = C$, since if we choose any $J' \notin C$, then $a_{J'} = b_{J'} \cdot a_{J'} \in \ker(C) \cdot a_{J'} \subseteq \ker(C)$ (where the inclusion holds because $\ker(C)$ is an ideal of A) and $a_{J'} \notin J$, i.e., $a_{J'} \in \ker(C)$ and $a_{J'} \notin J$ from which we conclude that $J' \notin h(I)$. Hence $h(I) \subseteq C$. … … . (*) We also know, by Lemma 2.1, that $h\left(\bigcap_{h(I')=C} I'\right) \subseteq C$; so that the element a_J (which was originally contained in I being its generator) is also contained in $\bigcap_{h(I')=C} I'$, i.e., $I \; \bigcap_{h(I')=C} I' = \mathcal{N}(C)$; so that by minimality of $\mathcal{N}(C)$, we conclude that $I = \mathcal{N}(C)$. Since $h(I) = h(\mathcal{N}(C)) = h\left(\bigcap_{h(I')=C} I'\right) \supseteq C$, we then have, alongside (*) above, that $h(I) = C$. We just set $j(C) = I$, thus completing the proof. □

The above result gives an efficient way of generating the hull-minimal ideals, $j(C)$, which exists in any hull-kernel regular algebra, A. Indeed, each $j(C)$ can be described, for any closed subset C of Prim(A), as

$$j(C) = \left\{ \sum_{k=1}^{n} x_{J_k} \cdot a_{J_k} \cdot y_{J_k} : x_{J_k}, y_{J_k} \in \mathbb{C}1 \oplus A, J_1, J_2, \dots, J_n \in \mathrm{Prim}(A) \backslash C \right\}.$$

We now discuss some of the structure of $j(C)$ in the general setting of an abstract hull-kernel regular algebra, A. We need the following lemma.

Lemma 2.4. *If* $\mathbb{P}$ *be the collection of all prime ideals of* A, *then Prim*$(A) \subseteq \mathbb{P}$.

Proof. Let $xy \in J = \ker(\pi) \in \mathrm{Prim}(A)$, where π is any algebraically irreducible representation of A. Then $\pi(x)\pi(y) = \pi(xy) = 0$; which implies $\pi(x) = 0$ or $\pi(y) = 0$, i.e., $x \in J$ or $y \in J$. □

Given any two closed subsets C_1, C_2 of Prim(A), it is clear that the following subsets of A are well-defined hull-minimal ideals: $j(C_1)$, $j(C_2)$, $j(C_1 \cap C_2)$,

$j(C_1 \cup C_2)$ and $j(C_1) \cap j(C_2)$. Naturally we may want to compare these ideals with one another. It is true that $j(C_k) \subseteq j(C_k) \cdot j(C_k) \subseteq j(C_k)$, $k = 1, 2$; so that (since $C_1 = C_1 \cup C_1$) we have $j(C_k) \cdot j(C_k) = j(C_k \cup C_k)$, $k = 1, 2$. However, it should be more general and appropriate to study the possibility of an equality between $j(C_1) \cdot j(C_2)$ and $j(C_1 \cup C_2)$ (or $j(C_1 \cap C_2)$) for any closed subsets C_1, C_2 of $\text{Prim}(A)$. The above lemma leads to an answer to this possibility.

Proposition 2.5. *Let A be a hull-kernel regular semi-simple algebra. If C_1, C_2 are any two closed subsets of* $\text{Prim}(A)$, *then $h(j(C_1) \cdot j(C_2)) \subseteq h(j(C_1 \cup C_2))$. In particular, $j(C_1 \cup C_2) \subseteq j(C_1) \cdot j(C_2)$.*

Proof.

$$\begin{aligned} h(j(C_1) \cdot j(C_2)) &:= \{J \in \text{Prim}(A) : J \supseteq j(C_1) \cdot j(C_2)\} \\ &= \{J \in \text{Prim}(A) : J \supseteq j(C_1) \text{ or } J \supseteq j(C_2)\} \text{ (by Lemma 2.4)} \\ &= \{J \in \text{Prim}(A) : J \supseteq j(C_1)\} \cup \{J \in \text{Prim}(A) : J \supseteq j(C_2)\} \\ &= h(j(C_1)) \cup h(j(C_2)) = C_1 \cup C_2 \text{ (being closed sets in } \text{Prim}(A)) \\ &\subseteq h(j(C_1 \cup C_2)). \end{aligned}$$

The second inclusion holds if we recall, from [6], that $h(E_1) \subseteq h(E_2) \Rightarrow E_2 \subseteq E_1$. □

Proposition 2.5 gives a very general relationship between the hull-minimal ideal $j(C_1 \cup C_2)$ and the ideal $j(C_1) \cdot j(C_2)$. It expresses in particular that $j(C_1) \cdot j(C_2) := \{ab : a \in j(C_1), b \in j(C_2)\}$ is generally larger than $j(C_1 \cup C_2)$. It is, however, known that if $ab = ba$ for $a \in j(C_1)$, $b \in j(C_2)$, then the size of $j(C_1) \cdot j(C_2)$ reduces considerably. One may want to know if this reduction could result to equality in Proposition 2.5. The following result answers the question in the affirmative.

Proposition 2.6. *Let A be an abelian hull-kernel regular algebra and let C_1 and C_2 be any two closed subsets of* $\text{Prim}(A)$. *Then $j(C_1) \cdot j(C_2) = j(C_1 \cup C_2)$.*

Proof. We only verify that $j(C_1) \cdot j(C_2) \subseteq j(C_1 \cup C_2)$. Now as A is hull-kernel regular, for each $k = 1, 2$ and any $J_k \in \text{Prim}(A)\backslash C_k$, there are $a_{J_k}, b_{J_k} \in A$ such that (i) $b_{J_k} \in \ker(C_k)$, $a_{J_k} \in A\backslash J_k$ and (ii) $b_{J_k} \cdot a_{J_k} = a_{J_k}$. From Theorem 2.3, we know that a_{J_1} generates $j(C_1)$ while a_{J_2} generates $j(C_2)$. In particular, we have $a_{J_1} \in j(C_1)$ and $a_{J_2} \in j(C_2)$ so that $a := a_{J_1} \cdot a_{J_2} \in j(C_1) \cdot j(C_2)$. It then follows that (i) $b := b_{J_1} \cdot b_{J_2} \in \ker(C_1) \cdot \ker(C_2) \subseteq \ker(C_1 \cup C_2)$, and, as A is abelian, we have (ii) $b \cdot a = b_{J_1} \cdot (b_{J_2} \cdot a_{J_1}) \cdot a_{J_2} = (b_{J_1} \cdot a_{J_1})(b_{J_2} \cdot a_{J_2}) = a_{J_1} a_{J_2}$ $= a$. Thus, by Definition 2.2, we conclude that $a \in j(C_1 \cup C_2)$ whenever $a \in j(C_1) \cdot j(C_2)$ as required. □

Thus the requirement of commutativity on A pays off and happens to be the exact condition that reduces the size of $j(C_1) \cdot j(C_2)$, and hence closes the gap between $j(C_1 \cup C_2)$ and $j(C_1) \cdot j(C_2)$. Now since the algebra in focus in this paper may not always be commutative as we have in the cases of all $\mathscr{C}_\tau(G)$ with $\tau \neq (1, 1)$, we may want to know if the equality in Proposition 2.6 is attained without the requirement of commutativity, or any other requirements at all on A. The answer is in the affirmative. However, before introducing the framework within which this answer holds we present the following lemma.

Lemma 2.7 ([6, p. 84]). *Let A be a hull-kernel regular algebra and let C_1 and C_2 be any two closed subsets of* $\text{Prim}(A)$ *such that $C_1 \subseteq C_2$. Then $j(C_1) \subseteq j(C_2)$.*

Proof. Let $a_J \in j(C_2)$. Then there exists $J \in \text{Prim}(A)\backslash C_2 (\subseteq \text{Prim}(A)\backslash C_1$, since $C_1 \subseteq C_2)$ such that $a_J, b_J \in A$ with $b_J \in \ker(C_2)$ $(\subseteq \ker(C_1)$, since $C_1 \subseteq C_2)$, $a_J \in A\backslash J$, and $b_J \cdot a_J = a_J$. Extracting the information that relates to C_1 above we have, (i) $b_J \in \ker(C_1)$, $a_J \in A\backslash J$ and (ii) $b_J \cdot a_J = a_J$, i.e., $a_J \in J(C_1)$. □

Since $(C_1 \backslash (C_1 \cap C_2)) \cup (C_2 \backslash (C_1 \cap C_2)) \subseteq (C_1 \cup C_2)$ the above lemma requires that $j(C_1 \cup C_2) \subseteq j((C_1 \backslash (C_1 \cap C_2)) \cup (C_2 \backslash (C_1 \cap C_2)))$, with equality whenever $C_1 \cap C_2 = \varnothing$. A more general requirement than $C_1 \cap C_2 = \varnothing$ is christened below.

Definition 2.8. Let A be as in Definition 2.2. The subsets C_1 and C_2 of $\text{Prim}(A)$ are said to be *hull-kernel separated* if there exist open subsets U_1 and U_2 in $\text{Prim}(A)$ such that $C_1 \subseteq U_1$, $C_2 \subseteq U_2$ and $U_1 \cap U_2 = \varnothing$.

The space Prim(A) is called a *hull-kernel* T_2-*space* whenever any two of its subsets are hull-kernel separated. The reader should compare Definition 2.8 with Definition 2.7.1 on p. 83 of [10]. The attending effect of this notion on the inclusion of Proposition 2.5 is the following.

Proposition 2.9. *Let A be a hull-kernel regular semi-simple algebra and let C_1 and C_2 be any two closed hull-kernel separated subsets of* Prim(A). *Then* $j(C_1) \cdot j(C_2) = j(C_1 \cup C_2)$.

Proof. Set $K_i = \text{Prim}(A) \backslash U_i$, for $i = 1, 2$ and U_i as in Definition 2.8. Then each K_i is closed in Prim(A) and since $U_1 \cap U_2 = \varnothing$, it follows that $U_1 \subset K_2$, so that $C_2 \subset U_1 \subset K_2$. In the same way, $C_2 \subset K_1$ while Prim(A) then becomes the union of the closed subsets K_1 and K_2. Now as $C_1 \cup C_2$ is closed in Prim(A), we know (from Proposition 2.3) that $j(C_1 \cup C_2)$ exists. Indeed, if $a_J \in j(C_1 \cup C_2)$ for any $J \notin C_1 \cup C_2$, then we can find $b_J \in \ker(C_1 \cup C_2)$ for which $a_J \notin J$ such that $b_J \cdot a_J = a_J$.

Clearly, $b_J \in \ker(C_1 \cup C_2) = \ker(C_1) \cap \ker(C_2)$ so that $b_J \in \ker(C_1)$ and $b_J \in \ker(C_2)$. By De Morgan's rule, we also have that $J \notin C_1$ and $J \notin C_2$. Since $j(K_1)$ is well-defined (as hull-minimal ideal of A) we have that for any $J \notin C_1$, we can choose $b_J \in \ker(K_1)$ and $a_J \notin J$ with $b_J \cdot a_J = a_J$, i.e., $a_J \in j(K_1)$ which immediately implies that $a_J \in j(C_2)$ (as $C_2 \subset K_1$). In the same way, $j(K_2)$ is well-defined and we then have that for any $J \notin C_2$, we can choose $b_J \in \ker(K_2)$ and $a_J \notin J$ with the requirement $b_J \cdot a_J = a_J$, i.e., $a_J \in j(K_2) \subseteq j(C_1)$. It then follows that each $j(C_i)$, $i = 1, 2$, is generated by elements of the form $b \cdot a_J \cdot a$, with $a, b \in A$, $J \notin C_i$ ... ... (1)

We claim that for any $a \in A$, $J_1 \notin C_1$, $J_2 \notin C_2$ we always have (by the choice of a_{J_1} and a_{J_2}) that $a_{J_1} \cdot a \cdot a_{J_2} \in j(C_1 \cup C_2)$... ... ... (2), where J_1 or $J_2 \notin C_1 \cup C_2$. Since otherwise if $J_1, J_2 \in C_1 \cup C_2$, then $a_{J_1} \cdot a \cdot a_{J_2} \in \ker(K_1)$ and $\in \ker(K_2)$ (from above) which immediately implies that $a_{J_1} \cdot a \cdot a_{J_2} \in \ker(K_1 \cup K_2)$ $= \ker(\text{Prim}(A)) = \{0\}$, as A is semi-simple. We conclude from (1) and (2) that if $c \in j(C_1) \cdot j(C_2)$, then $c \in j(C_1 \cup C_2)$. □

The above entails a rough estimate of the structure of the hull-minimal ideals corresponding to closed subsets of the structure space of a not-necessarily topologized algebra, A. However, the motivating example of a commutative Banach algebra ([5, p. 84]) suggests a robust theory of hull-minimal ideals when A is endowed with some of the common tools of functional analysis, such as norm, a pseudo-norm, and an inner-product. Thus from now on one could take the direction of a commutative Banach algebra, which has the well-understood Gelfand theory ([1]), or of a (group) C^*-algebra culminating in a detailed understanding of the Pedersen ideal ([9]) or of a Fréchet algebra (of functions on groups) a part of which has been considered by Ludwig [6]. We take the path of a Fréchet algebra (of functions on groups) and use the situation for a commutative Banach algebra as a guide. More precisely, we consider the Schwartz algebra of spherical functions on a connected semi-simple Lie group, with finite center and discuss the structure of its hull-minimal ideals. Our main results contained in Sections 3 and 4, stand in analogue to Ludwig's own in the case of the Schwartz algebra of functions on connected nilpotent Lie groups and reveal, in our own case, that the well-sought basis of these ideals are well-known objects in the modern theory of numbers; the cusp forms. We now take a step closer to this objective by first looking at the exact nature of the hull-minimal ideals in abstract Fréchet algebra, by seeking when this algebra is hull-kernel regular (cf. Theorem 2.3).

Let $\{p_k\}$ be a collection of seminorms that converts A into a Fréchet algebra with an involution. Define the map $e : A \to A$ as $e(a) = \sum_{k=1}^{\infty} \frac{a^k}{k!}$, $a \in A$; and call an element $a \in A$ *polynomially bounded* if for every $k \in \mathbb{N}$, there is a constant $c_k = c_k(a) > 0$ such that $p_k(e(i\lambda a)) \leq c_k(1 + |\lambda|)^{c_k}$ holds for all $\lambda \in \mathbb{R}$.

Restricting the notion of polynomially bounded elements to members of a Banach algebra, it follows that such elements must necessarily have real spectra ([6, p. 80]). We therefore seek a more general requirement that encompasses the real spectrum of an element of a Banach algebra for an element of a Fréchet algebra. Since an involutive Banach algebra in which the spectrum of every self-adjoint element is a subset of $\mathbb{R}$ is called *symmetric*, we consider next a *symmetric Fréchet algebra*.

Definition 2.10. A Fréchet algebra A is said to be *symmetric* if it admits a continuous involution and if there exists a continuous *-homomorphism, σ, of A into a C^*-algebra, C, such that $\mathrm{spec}_A(a) = \mathrm{spec}_C(\sigma(a))$ for every $a \in A$. (Here $\mathrm{spec}_A(a)$ denotes the spectrum of an element a in A.)

The following results, whose proofs can be found in [6], have been generalized to a quasi-symmetric Fréchet algebra in an upcoming paper by the authors and they tell us exactly when a Fréchet algebra admits hull-minimal ideals.

Lemma 2.11 ([6, p. 81]). *Every algebraically irreducible representation of a symmetric Fréchet algebra A is equivalent to a submodule of a topologically irreducible representation of A.*

Using this lemma one can establish that the type of Fréchet algebra that admits hull-minimal ideals are christened as follows.

Definition 2.12. An involutive Fréchet algebra A is said to be *polynomially bounded* if the set A_0 of self-adjoint polynomially bounded elements of A is dense in the real subspace A_h of hermitian elements of A.

It then follows that an involutive Schwartz algebra is a polynomially bounded Fréchet algebra and that the sufficient conditions on a Fréchet algebra to admit hull-minimal ideals are contained in the following result.

Lemma 2.13 ([6, p. 81]). *Every semi-simple symmetric polynomially bounded Fréchet algebra is hull-kernel regular.*

It then follows, from Theorem 2.3, that if A is a semi-simple polynomially bounded Fréchet algebra and C is any closed subset of $\mathrm{Prim}(A)$, then the hull-minimal ideal, $j(C)$, exists and is generated by the elements a_J, $J \notin C$. It therefore means that when in possession of a Fréchet algebra whose hull-minimal ideals are to be studied we must first and foremost establish that it is semi-simple, symmetric and polynomially bounded. Thus our first task in the next section is to establish that the semi-simple polynomially bounded Harish-Chandra algebra, $\mathscr{C}(G)$, is symmetric.

3. Hull-minimal Ideals in $\mathscr{C}(G)$

Define the *minimal regular norm*, $\|\cdot\|$, on $L^1(G)$ as $\|f\| = \sup\|\pi(f)\|_1$, where $\|\cdot\|_1$ is the L^1-norm on G and π runs through the set of non-degenerate

$*$-representations of $L^1(G)$. The completion of $L^1(G)$ with respect to $\|\cdot\|$ is a C^*-algebra often called the *group C^*-algebra* of G and denoted as $C^*(G)$. The following result is well-known.

Theorem 3.1 (cf. [13, p. 161]). *$\mathscr{C}(G)$ is dense in $C^*(G)$.*

We now have the following:

Proposition 3.2. *The identity map is a continuous $*$-homomorphism of $\mathscr{C}(G)$ into $C^*(G)$.*

Proof. Since both $\mathscr{C}(G)$ and $C^*(G)$ are involutive algebras under the usual involution $f \mapsto f^*$, given as $f^*(x) = \overline{f(x^{-1})}$, the identity map is then a $*$-homomorphism of $\mathscr{C}(G)$ into $C^*(G)$ from Theorem 3.1. For its continuity, let $\mathcal{O}$ be an open set in $L^2(G)$ defined by the seminorms $f \mapsto \| f \|_{a,b:r}$ $\left(:= \left(\int_G |(afb)(x)|^2 (1+\sigma(x))^{2r} dx \right)^{\frac{1}{2}} \right)$. Since these seminorms induce the topology of $\mathscr{C}(G)$ ([11, p. 348]) it follows that $\mathcal{O}$ is also an open set in $\mathscr{C}(G)$. Hence the topology on $\mathscr{C}(G)$ is in particular stronger than that on $C^*(G)$. □

Another continuous $*$-homomorphism of $\mathscr{C}(G)$ into $C^*(G)$ is $f \mapsto \| f \|_{a,b:r}$. Our first major result on $\mathscr{C}(G)$ is then the following:

Theorem 3.3. *The Harish-Chandra Schwartz algebra $\mathscr{C}(G)$ is a symmetric Fréchet algebra.*

Proof. The inequalities $\mu_{a,b:r}(f^* * f) \leq c_0 \mu_{a,1:r}(f^*) \mu_{1,b:r+r_0+1}(f) \leq c_0 \mu_{a,1:r}(f) \mu_{1,b:r+r_0+1}(f)$, for every $f \in \mathscr{C}(G)$, $a, b \in U(\mathfrak{g}_{\mathbb{C}})$, imply that the involution $f \mapsto f^*$ on $\mathscr{C}(G)$ is continuous. Using Proposition 3.2 leads to the assertion. □

The importance of the last result is enormous if we recall the fact that $L^1(G)$ is not symmetric in the sense of [2]. In fact armed with the conclusion of Lemma 2.13, we can state another result of this section.

Proposition 3.4. *The Harish-Chandra Schwartz algebra $\mathscr{C}(G)$ admits hull-minimal ideals, i.e., for any closed subset C of* Prim$(\mathscr{C}(G))$, *the hull-minimal ideal $j(C)$ exists and is generated by the elements a_J, $J \not\in C$.*

Even though the Harish-Chandra Schwartz algebra is not commutative, our choice of closed subsets in Prim$(\mathscr{C}(G))$ can still lead to a fine structural relationship between the hull-minimal ideals of $\mathscr{C}(G)$. Indeed, we have the following:

Corollary 3.5. *If* C_1 *and* C_2 *are any two closed hull-kernel separated subsets of* Prim$(\mathscr{C}(G))$, *then* $j(C_1) \cdot j(C_2) = j(C_1 \cup C_2)$.

Proof. See Proposition 2.9. □

It is, however, clear at this juncture that even though we have been considering a specific hull-kernel regular algebra in this section, as against the very abstract consideration in Section 2, the results of Proposition 3.4 and Corollary 3.5 are still not explicit enough. We would like to have a more explicit description of $j(C)$ in Prim$(\mathscr{C}(G))$ than is contained in Proposition 3.4, and thus study more structural relationships between them using this concrete description. Indeed, this is the motivation behind [6], and in the present paper, we focus on $\mathscr{C}(G)$ and seek to give an explicit description of the basis elements of the hull-minimal ideals in Prim$(\mathscr{C}(G))$. We start with the following major property of $\mathfrak{Z}$-finite K-finite functions.

Lemma 3.6 ([11, p. 352]). *Let f be a $\mathfrak{Z}$-finite K-finite function in $C^{\infty}(G)$ and let $\mathscr{H}$ be the set of all $h \in C_c^{\infty}(G)$ such that $h(kxk^{-1}) = h(x)$, for all $k \in K$, $x \in G$, with support arbitrarily close to 1. Then there exists $h \in \mathscr{H}$ for which $h * f = f$.*

If we denote the subspace of $C_c^{\infty}(G)$ consisting of functions with support arbitrarily close to 1 by $C_{c,1}^{\infty}(G)$, we refer to members of $\mathscr{H}$ as *K-central functions* in $C_{c,1}^{\infty}(G)$, and note that in this terminology, the above lemma says that every

$\mathfrak{Z}$-finite K-finite functions in $C^\infty(G)$ is expressible as its convolution with some K-central functions in $C_{c,1}^\infty(G)$. We define the Harish-Chandra Schwartz algebra of spherical functions as $\mathscr{C}(G // K) = \{f \in \mathscr{C}(G) : f(k_1 x k_2) = f(x), k_1, k_2 \in K, x \in G\}$ and note that $\mathscr{H} \subset \mathscr{C}(G // K)$. Since it is known that every $\mathfrak{Z}$-finite K-finite f in $C^\infty(G) \cap L^2(G)$ satisfies the weak inequality and is automatically contained in $\mathscr{C}(G)$ ([12, p. 369]), we have the following upliftment of Lemma 3.6 to the level of the Harish-Chandra Schwartz algebra.

Corollary 3.7. *Every (non-zero) $\mathfrak{Z}$-finite K-finite function in $\mathscr{C}(G)$ is expressible as its convolution with some member of $\mathscr{H}$.*

It is, however, known that the $\mathfrak{Z}$-finite K-finite functions in $\mathscr{C}(G)$ are explicitly expressed as the linear combination of the distributional characters of the discrete series of representations of G ([12, p. 398]). This means that every $\mathfrak{Z}$-finite K-finite function in $\mathscr{C}(G)$ is in the linear span of the K-finite matrix coefficients of the discrete series of representations of G, which according to Harish-Chandra [3] is exactly the space ${}^\circ\mathscr{C}(G) := \left\{f \in \mathscr{C}(G) : \int_N f(xn)dn = 0, x \in G = KAN\right\}$ of cusp forms. Since the space ${}^\circ\mathscr{C}(G)$ of cusp forms is orthogonal to the matrix elements of the (unitary) principal series of representations of G and since the discrete series of representations are never spherical ([12, p. 272]) (so that the members of $\mathscr{C}(G // K)$ are never the matrix-elements of the discrete series of representations of G), it follows that ${}^\circ\mathscr{C}(G) \cap \mathscr{C}(G // K) = {}^\circ\mathscr{C}(G // K) = \{0\}$, and hence that $\mathscr{H} \cap {}^\circ\mathscr{C}(G // K) = \{0\}$. This explains that the members of $\mathscr{H}$ are associated only to the (unitary) principal series of the representations of G.

We are also interested in the collection of representations of G that kill members of $\mathscr{H}$ now defined below.

Definition 3.8. A representation π of G is called *singular* whenever $f \in \ker(\pi)$ for some $f \in C_c(G)$. We denote the set of singular representations of G as $\hat{G}_{\text{sing}}$.

If $\mathscr{H}^*$ denotes the set of representations of G whose kernel spans $\mathscr{H}$, then $\mathscr{H}^* \subset \hat{G}_{\text{sing}}$. We also have the following.

Theorem 3.9. *The set* $\hat{G}_{\text{sing}}$ *is closed in* $\hat{G}$.

Proof. Let $\{\pi_\alpha\}_{\alpha\in\Omega}$ be a sequence of representations in $\hat{G}_{\text{sing}}$ and let $\lim_{\alpha\to\infty}\pi_\alpha = \pi_\circ$ in the topology induced on $\hat{G}_{\text{sing}}$ as a subset of $\hat{G} \simeq \text{Prim}(\mathscr{C}(G))$ ([8, p. 237]). Then $\pi_\circ$ is a representation of G (due to the continuity of each π_α) and given any $\varepsilon > 0$, there are integers $N_1 = N_1(\varepsilon) > 0$ and $N_2 = N_2(\varepsilon) > 0$ such that $\| \pi_\alpha(f) \| < \frac{\varepsilon}{2}$, for all $\alpha > N_1$ and $\| \pi_\circ(f) - \pi_\alpha(f) \| < \frac{\varepsilon}{2}$ for all $\alpha > N_2$, and some $f \in C_c(G)$. Set $N = \min\{N_1, N2\}$, then for all $\alpha > N$, we have that $\| \pi_\circ(f) \|$ $= \| (\pi_0(f) - \pi_\alpha(f)) + \pi_\alpha(f) \| \leq \| \pi_\circ(f) - \pi_\alpha(f) \| + \| \pi_\alpha(f) \| < \varepsilon$, i.e., $\pi_\circ \in G_{\text{sing}}$.

□

The above theorem assures us that $j(\hat{G}_{\text{sing}})$ exists in $\mathscr{C}(G)$. Now let $\hat{G}_d$ be subset of $\hat{G}$ consisting of the discrete series of representations of G. Then the following holds.

Proposition 3.10. $\hat{G}_{\text{sing}} \cap \hat{G}_d = \varnothing$.

Proof. Some of the principal series of representation of G are contained in $\hat{G}_{\text{sing}}$ (because ${}^\circ\mathscr{H} = \{0\}$) and since $\hat{G}_{\text{sing}}$ is closed in $\hat{G}$, it cannot contain a discrete series of representation of G. □

The last result of this section is then the following.

Theorem 3.11. *Let C be a closed subset of $\hat{G}$ containing $\mathscr{H}^*$ as a dense subset. Then the hull-minimal ideal, $j(C)$ of $\mathscr{C}(G)$, corresponding to C is the linear span of the $\mathfrak{Z}$-finite K-finite functions in $\mathscr{C}(G)$.*

Proof. Let C be as in the hypothesis. Then we can choose $h \in \mathscr{H}^*$ such that $h \in \ker(\pi)$, for every $\pi \in \hat{G}$. In particular, $h \in \ker(\pi)$ for every $\pi \in \mathscr{H}^*$. Since $\mathscr{H}^*$ is dense in C, we conclude that $h \in \ker(\pi)$ for every $\pi \in C$, i.e., $h \in \ker(C)$. Since we already know, from Lemma 3.6, that $h * f = f$ for $\mathfrak{Z}$-finite K-finite functions in $\mathscr{C}(G)$, the assertion holds. □

The hull-minimal ideals, $j(C)$, corresponding to a closed subset C of $\hat{G}$ and containing $\mathscr{H}^*$ (as in Theorem 3.11 above) shall be called the *type-I hull-minimal ideals* of $\mathscr{C}(G)$. This is due to other types considered in [6] not included in this work.

Corollary 3.12. *Every type-I hull-minimal ideal in* $\mathscr{C}(G)$ *is a* $\mathfrak{Z}(\mathfrak{g})$*-module of* $\mathscr{C}(G)$.

Proof. It is well known (cf. [11]) that $\mathscr{C}(G)$ is a $U(\mathfrak{g})$-module. Now since the basis elements of the hull-minimal ideals of $\mathscr{C}(G)$ are $\mathfrak{Z}$-finite K-finite functions in $\mathscr{C}(G)$ which are essentially the linear span of the matrix-coefficients of the discrete series of representations of G ([12, p. 399]) it follows from the transformation satisfied by these matrix coefficients under the action of $\mathfrak{Z}(\mathfrak{g})$ that the result holds. □

4. Hull-minimal Ideals in $\mathscr{C}(G // K)$ and $\mathscr{C}_\tau(G)$

Let $\tau = (\tau_1, \tau_2)$ be a double representation of K on a finite dimensional vector space, W. By a *τ-spherical function* we mean a continuous function $f : G \to W$ such that $f(k_1 x k_2) = \tau_1(k_1) f(x) \tau_2(k_2)$ and which is also an eigenfunction for a suitable algebra of operators on G. We denote the Schwartz algebra consisting of τ-spherical functions on G by $\mathscr{C}_\tau(G)$. Clearly $\mathscr{C}_{(1,1)}(G) = \mathscr{C}(G // K)$ and is a commutative subalgebra of $\mathscr{C}(G)$. However, the Harish-Chandra philosophy of cusp forms gives the following.

Proposition 4.1. $\mathscr{C}(G // K)$ *does not admit type-I hull-minimal ideals.*

Proof. It follows from the use of ${}^\circ\mathscr{C}(G // K) = \{0\}$ in Theorem 3.11. □

However, $\mathscr{C}(G // K)$ may have other types of hull-minimal ideals (see [6]). Also, what is not available in the subalgebra $\mathscr{C}(G // K)$ is contained in $\mathscr{C}_\tau(G)$ for $\tau \neq (1, 1)$ since due to the relationship of the Eisenstein integrals on G with the discrete series representations of G we have that ${}^\circ\mathscr{C}_\tau(G) \neq \{0\}$ for all $\tau \neq (1, 1)$. Indeed, Theorem 3.11 holds for $\mathscr{C}(G)$ replaced with $\mathscr{C}_\tau(G)$, $\tau \neq (1, 1)$.

Remarks 4.2. (a) It is clear from the above analysis that the study of the hull-minimal ideals in both $\mathscr{C}(G)$ and $\mathscr{C}_\tau(G)$, $\tau \neq (1, 1)$ does not necessarily require the direct involvement of the Fourier transform of the functions concerned as unavoidably used by Ludwig [6] in the connected nilpotent Lie group case. This is simply because of the well-known developed theory of the $\mathfrak{Z}$-finite K-finite functions in $\mathscr{C}(G)$. Thus the result of Theorem 3.11 suggests that the Ludwig's functions in the Schwartz algebra $S(G)$, of a connected nilpotent Lie group G that serve as the basis of his hull-minimal ideals are analogues of our present $\mathfrak{Z}$-finite K-finite functions in $\mathscr{C}(G)$, and their properties, such as their differential equations and hence their explicit representations, could be sought in this light.

(b) In order to have a good grasp on the hull-minimal ideals in $\mathscr{C}_\tau(G)$ it is pertinent to have a direct involvement of the Eisenstein integrals and the discrete series of representations of G attached to these integrals. This will be the subject of another paper.

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