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On a Four-Parameter Type II Generalized Half Logistic Distribution

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Abstract

In this paper, we derived a four-parameter type II generalized half logistic distribution. The cumulative distribution function, the survival and hazard function, the moments, the median, the 100p-percentage point and the mode of the generalized distribution are established. Some theorems that characterized the probability distribution function are stated and proved.

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1 Introduction

A random variable X is said to have an half logistic distribution if its probability density function is

$$f_X(x) = \frac{2exp(x)}{(1+exp(x))^2}, \quad x > 0. \tag{1.1}$$

Balakrishnan (1985) established some recurrence relations for the moments and product moments of order statistics for half logistic distribution. If a location parameter μ and a

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scale parameter σ are introduced in the equation (1.1), the density of the random variable X becomes

$$f_X(x;\mu,\sigma)=\frac{2exp(\frac{x-\mu}{\sigma})}{\sigma(1+exp(\frac{x-\mu}{\sigma}))^2},\quad x>0,\ \mu\geq 0,\ \sigma>0. \tag{1.2}$$

Balakrishnan and Puthenpura (1986) obtained the best linear unbiased estimates of the location and scale parameters μ and σ respectively of the half logistic distribution through linear functions of order statistics and they also tabulated the values of the variance and covariance of these estimators. Balakrishnan and Wong (1991) obtained approximate maximum likelihood estimates of the location and scale parameters of the half logistic distribution with Type-II Right-Censoring. Olapade (2003) stated and proved some theorems that characterized the half logistic distribution.

2 Four-parameter type II generalized half logistic distribution

Olapade (2005) obtained a generalized form of the logistic distribution which contains four parameters as

$$f_X(x;\mu,\sigma,\lambda,p)=\frac{p\lambda}{\sigma}\frac{e^{-p(\frac{x_i-\mu}{\sigma})}}{[\lambda+e^{-(\frac{x_i-\mu}{\sigma})}]^{(p+1)}},\quad -\infty<x<\infty, \tag{2.0}$$

where μ is the location parameter, σ is the scale parameter, p is the shape parameter and λ is a shift parameter. In this research, we want to obtain a generalized half logistic form of the generalized logistic distribution in equation (2.0) which we shall call a four-parameter type II generalized half logistic distribution and study its theories and properties. Let us consider the theorem below.

Theorem 1: Suppose a continuously distributed random variable Y has probability density function $f_Y(y)$, then the random variable $X=\ln(\lambda e^{-Y}/(2-e^{-Y}))$ has a generalized half logistic distribution if Y has an exponential probability density function with parameter b where $\lambda>0$ is a constant.

Proof: If Y is exponentially distributed with parameter b, then the probability density function of Y is

$$f_Y(y)=be^{-by},\quad x>0,\quad b>0.$$

Then $x=\ln(\lambda e^{-y}/(2-e^{-y}))$ implies that

$$y=\ln(\frac{\lambda+e^x}{2e^x})\ \text{and}\ \frac{dy}{dx}=\frac{-\lambda}{\lambda+e^x},$$

$$f_X(x) = K|\frac{dy}{dx}|f_Y(x) = \frac{Ke^{bx}}{(\lambda + e^x)^{b+1}}, \tag{2.1}$$

where $K = b\lambda(1-(\lambda+1)^{-b})^{-1}$. Then

$$f_X(x; b, \lambda) = \frac{b\lambda}{(1-(\lambda+1)^{-b})} \frac{e^{bx}}{(\lambda + e^x)^{b+1}}, \; x > 0, \; b > 0, \; \lambda > 0. \tag{2.2}$$

If we introduce the location parameter μ and the scale parameter σ in equation (2.2), we have

$$f_X(x; \mu, \sigma, b, \lambda) = \frac{b\lambda}{\sigma(1-(\lambda+1)^{-b})} \frac{e^{b(\frac{x-\mu}{\sigma})}}{(\lambda + e^{(\frac{x-\mu}{\sigma})})^{b+1}}, \; x > 0, \; \mu > 0, \; \sigma > 0, \; b > 0, \; \lambda > 0. \tag{2.3}$$

This probability density function in equation (2.3) is what we call a four-parameter type II generalized half logistic distribution. It gives the half logistic version of the negatively skewed extended generalized logistic distribution in equation (2.0). In the rest of this paper, we shall assume that $\mu = 0$ and $\sigma = 1$ without loss of generality.

3 The cumulative distribution function, the survival function and the hazard function of the four-parameter type II generalized half logistic distribution

The cumulative distribution function of the four-parameter type II generalized half logistic distribution is obtained from equation (2.2) as

$$\begin{aligned} F_X(x; b, \lambda) &= \int_0^x f_X(t)dt = \frac{b\lambda}{(1-(\lambda+1)^{-b})} \int_0^\infty \frac{e^{bt}}{(\lambda + e^t)^{b+1}} dt \\ &= \frac{1}{(1-(\lambda+1)^{-b})} [\frac{e^{bx}}{(\lambda + e^x)^b} - \frac{1}{(\lambda+1)^b}]. \end{aligned} \tag{3.1}$$

As the values of $F_X(x; b, \lambda)$ depends on the values of b, λ and x, the probability that a four-parameter type II generalized half logistic random variable X lies in an interval (α_1, α_2) is obtained as

$$\begin{aligned} pr(\alpha_1 < X < \alpha_2) &= F_X(\alpha_2; b, \lambda) - F_X(\alpha_1; b, \lambda) \\ &= \frac{1}{(1-(\lambda+1)^{-b})} [\frac{e^{b\alpha_2}}{(\lambda + e^{\alpha_2})^b} - \frac{e^{b\alpha_1}}{(\lambda + e^{\alpha_1})^b}] \end{aligned} \tag{3.2}$$

for any real value of b, λ and any given interval (α_1, α_2).

If X is the lifetime of an object, then the survival function $S(x)$ of X is

$$S(x) = pr(X > x) = 1 - \frac{1}{(1-(\lambda+1)^{-b})}[\frac{e^{bx}}{(\lambda+e^{x})^{b}} - \frac{1}{(\lambda+1)^{b}}]. \quad (3.3)$$

This is the probability that an object whose life time is a random variable X that follows the four-parameter type II generalized half logistics distribution will survive beyond time x.

The hazard function of the random variable X is $H(x) = f(x)/(1 - F(x))$, then if X follows the four-parameter type II generalized half logistics distribution, its hazard function is

$$H(x) = \frac{\lambda b e^{bx}}{(\lambda+e^{x})^{b+1}[(1-(\lambda+1)^{-b}) - \frac{e^{bx}}{(\lambda+e^{x})^{b+1}} + \frac{1}{(\lambda+1)^{b}}]}. \quad (3.4)$$

4 Moments of the Four-parameter Type II Generalized Half Logistic Distribution

Considering the four-parameter type II generalized half logistic distribution function $f_X(x; b, \lambda)$ given in equation (2.2). The n^{th} moment of X is

$$E[X^n] = \frac{b\lambda}{(1-(\lambda+1)^{-b})}\int_0^\infty \frac{x^n e^{bx}}{(\lambda+e^{x})^{b+1}}dx. \quad (4.1)$$

The tables below shows a tabulated value of $E[X^n]$ for $b = 0.25, 0.50, 0.75, 1.00$, $1.50, 2.00, 2.50, 3.00, 3.50$ and 4.00 when $\lambda = 1.5, 2.0, 2.5$ and 3.0. These values can be used to compute the mean, variance, skewness and kurtosis for the four-parameter type II generalized half logistic distribution using the following relations:

$$\mu_1 = \nu_1$$

$$\mu_2 = \nu_2 - \nu_1^2$$

$$\mu_3 = \nu_3 - 3\nu_2\nu_1 + 2\nu_1^3$$

$$\mu_4 = \nu_4 - 4\nu_3\nu_1 + 6\nu_2\nu_1^2 - 3\nu_1^4 \quad (4.2)$$

where ν_i is the i^{th} moment $E[X^i]$ and μ_1 =the mean, μ_2=the variance, skewness $\beta_1 = \mu_3^2/\mu_2^3$ and the measure of kurtosis $\beta_2 = \mu_4/\mu_2^2$.

Tables of moments of the four-parameter type II generalized half logistic distribution.

Table 4.1 : $\lambda = 1.5$

b	$E[X]$	$E[X^2]$	$E[X^3]$	$E[X^4]$
0.2500	1.3195	3.0678	9.9944	41.8081
0.5000	1.3880	3.3122	10.9450	46.1469
0.7500	1.4573	3.5648	11.9407	50.7279
1.0000	1.5271	3.8245	12.9782	55.5383
1.5000	1.6670	4.3614	15.1629	65.7858
2.0000	1.8081	4.9134	17.4657	76.7509
2.5000	1.9395	5.4721	19.8519	88.2819
3.0000	2.0687	6.0295	22.2886	100.2277
3.5000	2.1915	6.5792	24.7467	112.4488
4.0000	2.3073	7.1166	27.2021	124.8248

Table 4.2 : $\lambda = 2.0$

b	$E[X]$	$E[X^2]$	$E[X^3]$	$E[X^4]$
0.2500	1.3911	3.3370	11.0772	46.8552
0.5000	1.4759	3.6504	12.3250	52.6342
0.7500	1.5617	3.9754	13.6396	58.7819
1.0000	1.6479	4.3102	15.0144	65.2719
1.5000	1.8192	5.0004	17.9136	79.1518
2.0000	1.9859	5.7045	20.9587	93.9971
2.5000	2.1451	6.4079	24.0874	109.5211
3.0000	2.2947	7.0988	27.2451	125.4596
3.5000	2.4338	7.7689	30.3882	141.5906
4.0000	2.5623	8.4129	33.4853	157.7417

Table 4.3 : $\lambda = 2.5$

b	$E[X]$	$E[X^2]$	$E[X^3]$	$E[X^4]$
0.2500	1.4536	3.5812	12.0849	51.6286
0.5000	1.5530	3.9593	13.6204	58.8323
0.7500	1.6534	4.3523	15.2448	66.5382
1.0000	1.7539	4.7569	16.9467	74.9468
1.5000	1.9519	5.5880	20.5338	92.1831
2.0000	2.1419	6.4280	24.2802	110.8241
2.5000	2.3199	7.2563	28.0928	130.1794
3.0000	2.4841	8.0582	31.8970	149.8689
3.5000	2.6341	8.8247	35.6390	169.6008
4.0000	2.7702	9.5514	39.2851	189.1714

Table 4.4 : $\lambda = 3.0$

b	$E[X]$	$E[X^2]$	$E[X^3]$	$E[X^4]$
0.2500	1.5093	3.8058	13.0315	56.1776
0.5000	1.6218	4.2449	14.8460	64.7887
0.7500	1.7352	4.7017	16.7711	74.0392
1.0000	1.8484	5.1716	18.7901	83.8595
1.5000	2.0696	6.1327	23.0373	104.8894
2.0000	2.2787	7.0947	27.4440	127.2160
2.5000	2.4715	8.0317	31.8863	150.2264
3.0000	2.6464	8.9270	36.2724	173.4324
3.5000	2.8036	9.7724	40.5434	196.4907
4.0000	2.9446	10.5655	44.6677	219.1862

5 The median of the four-parameter type II generalized half logistic distribution

The median of a probability density function $f(x)$ is a point x_m on the real line such that $\int_{-\infty}^{x_m} f(x)dx = \frac{1}{2}$, that is $F(x_m) = 0.5$. For the four-parameter type II generalized half logistic distribution, it implies that

$$\frac{1}{(1-(\lambda+1)^{-b})}[\frac{e^{bx}}{(\lambda+e^{x})^{b}} - \frac{1}{(\lambda+1)^{b}}] = 1/2, \tag{5.1}$$

$$\Rightarrow \frac{e^{x}}{1+e^{x}} = \sqrt[b]{\frac{1}{2}(1+(\lambda+1)^{-b})}.$$

Therefore,

$$x_m = \ln(\frac{\lambda\sqrt[b]{\frac{1}{2}(1+(\lambda+1)^{-b})}}{1-\sqrt[b]{\frac{1}{2}(1+(\lambda+1)^{-b})}}). \tag{5.2}$$

This value of the median reduces to ln3 when $b = 1,\ \lambda = 1$ as the median of standard half logistic distribution.

6 The $100p$-percentage point of the four-parameter type II generalized half logistic distribution

For this distribution the 100p-percentage point is obtained from the equation

$$F(x_p) = \frac{1}{(1-(\lambda+1)^{-b})}[\frac{e^{bx_p}}{(\lambda+e^{x_p})^{b}} - \frac{1}{(\lambda+1)^{b}}] = p, \quad 0 \le p \le 1. \tag{6.1}$$

We solved for x_p as

$$x_p = \ln[\frac{\lambda\sqrt[b]{p-(p-1)(\lambda+1)^{-b}}}{1-\sqrt[b]{p-(p-1)(\lambda+1)^{-b}}}]. \tag{6.2}$$

This reduces to ln3 when $b = 1,\ \lambda = 1$ and $p = 0.5$ which corresponds to the value of median of the standard half logistic distribution. The knowledge of x_p for several values of p provides an indication of how the unit probability mass is distributed over the real line.

7 The mode of the four-parameter type II generalized half logistic distribution

For this distribution the mode is obtained by differentiating the probability density function in equation (2.2) with respect to x, we have

$$f_X'(x) = \frac{b\lambda}{1-(\lambda+1)^{-b}}[\frac{b\lambda e^{bx} - e^{x(b+1)}}{(\lambda+e^x)^{b+2}}]. \tag{7.1}$$

Equating the derivative to zero gives and solve gives $x = \ln b\lambda$, which is the mode of the distribution as confirmed when $b = 1$, $\lambda = 1$ for standard half logistic distribution. Further confirmation of the mode of the distribution could be obtained by taking the second derivative of the probability density function and check the sign of its value when $x = \ln b\lambda$, which is found to be negative.

8 Some theorems that characterize the four-parameter type II generalized half logistic distribution

Here we state and prove two theorems that characterize this distribution.

Theorem 8.1: Suppose a continuous random variable Y has a uniform distribution over a unit range $(0,1)$, then the random variable

$$X = \ln(\frac{\lambda\sqrt[b]{Y}}{2-\sqrt[b]{Y}})$$

has a four-parameter type II generalized half logistic distribution with parameters b and λ.

Proof: If a random variable Y has a uniform distribution over a unit range $(0,1)$, then

$$f_Y(y) = 1, \quad 0 < y < 1.$$

Also, $x = \ln(\frac{\lambda\sqrt[b]{y}}{2-\sqrt[b]{y}})$ implies that

$$y = \frac{2^b e^{bx}}{(\lambda+e^x)^b} \quad \text{and} \quad \frac{dy}{dx} = \frac{2^b b\lambda e^{bx}}{(\lambda+e^x)^{b+1}}$$

From the theory of mathematical statistics,

$$f_X(x) = |\frac{dy}{dx}|f_Y(x)$$

Since $f_Y(y) = 1$, then

$$f_X(x) = \frac{Q2^b b\lambda e^{bx}}{(\lambda + e^x)^{b+1}}, \tag{8.1}$$

where Q is a normalizing constant. The value of Q that makes $f_X(x)$ a probability density function is $Q = (2^b(1 - (\lambda + 1)^{-b}))^{-1}$. Therefore,

$$f_X(x; b, \lambda) = \frac{b\lambda}{(1 - (\lambda + 1)^{-b})} \frac{e^{bx}}{(\lambda + e^x)^{b+1}}. \tag{8.2}$$

This completes the prove.

Theorem 8.2: The random variable X follows a four-parameter type II generalized half logistic distribution with parameters b, λ if and only if the density function f satisfies the homogeneous differential equation

$$(e^x + \lambda)f' + (e^x - b\lambda)f = 0, \tag{8.3}$$

(prime denotes differentiation).

Proof: Suppose X is a four-parameter type II generalized half logistic distribution random variable, then

$$f_X(x) = \frac{b\lambda}{(1 - (\lambda + 1)^{-b})} \frac{e^{bx}}{(\lambda + e^x)^{b+1}}.$$

By differentiating $f_X(x)$ with respect to x, we have

$$f'_X(x) = \frac{b\lambda}{1 - (\lambda + 1)^{-b}} [\frac{b\lambda e^{bx} - e^{bx} e^x)}{(\lambda + e^x)^{b+2}}].$$

By substituting $f(x)$ and $f'(x)$ in the differential equation (8.3), the equation is satisfied.

Conversely, we assume that f satisfies equation (8.3) and separate the variables and then integrate, we have

$$\int \frac{f'}{f} dx = \int (\frac{b\lambda - e^x}{\lambda + e^x}) dx \tag{8.4}$$

$$\ln f = b\lambda \int \frac{dx}{\lambda + e^x} - \int \frac{e^x}{\lambda + e^x} dx$$

$$\ln f = b\ln(\frac{e^x}{\lambda + e^x}) - \ln(\lambda + e^x) + \ln c$$

$$\ln f = \ln(\frac{e^{bx}}{(\lambda + e^x)^{b+1}}) + \ln c. \tag{8.5}$$

Therefore,

$$f = \frac{Ce^{bx}}{(\lambda + e^x)^{b+1}}, \quad 0 < x < \infty, \quad b > 0, \ \lambda > 0, \tag{8.6}$$

where C is a constant. The value of C that makes f a probability density function is $C = \frac{b\lambda}{(1-(\lambda+1)^{-b})}$. This completes the prove.

9 Conclusion

We have established the probability density function of a four-parameter type II generalized half logistic distribution with its cumulative distribution function, survival function and hazard function. The moments $E[X^i]$ for $i = 1, 2, 3, 4$ have been tabulated for some values of parameters b and λ. The median, the $100p$-percentage point and the mode of the distribution were presented. Two theorems that characterized the distribution were stated and proved. Further generalization of the distribution which contains four parameters to a five-parameter type II generalized half logistic distribution is currently under study.

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