ON A SIX-PARAMETER GENERALIZED BURR XII DISTRIBUTION

A. K. Olapade

Department of Statistics and Actuarial Science
University of Stellenbosch
Stellenbosch, South Africa

Correspondence: Department of Mathematics, Obafemi Awolowo University, Ile-Ife, Nigeria.
E-mails: akolapad@oauife.edu.ng; olapade@sun.ac.za

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ABSTRACT

In this paper, we derive a probability density function that generalizes the Burr XII distribution. The cumulative distribution function and the n-th moment of the generalized distribution are obtained while the distribution of some order statistics of the distribution is established. A theorem that relates the new distribution to another statistical distribution is established.

KEYWORDS: Generalized Burr XII distribution, Gamma distribution, Exponential distribution, Order Statistics.

2000 Mathematics Subject Classifications: Primary 62E15; Secondary 62E10.

INTRODUCTION

A continuous random variable X is said to follow a Burr XII distribution if its cumulative distribution function is given by

$$F(x) = 1 - (1 + \theta x^p)^{-m}, \quad 0 \le x < \infty, \ m > 0, \ \theta > 0, \ p > 0; \tag{1}$$

and its probability density function is given by

$$f(x) = \frac{m\theta p x^{p-1}}{(1 + \theta x^p)^{m+1}}, \quad 0 \le x < \infty, \ m > 0, \ \theta > 0, \ p > 0. \tag{2}$$

If the location parameter μ and the scale parameter σ are introduced in the Equation (2), we have

$$f(x; \mu, \sigma, m, \theta, p) = \frac{m\theta p \left(\frac{x-\mu}{\sigma}\right)^{p-1}}{\sigma \left[1 + \theta \left(\frac{x-\mu}{\sigma}\right)^p\right]^{m+1}}, \quad 0 \le \mu < \infty, \ \sigma > 0. \tag{3}$$

Correspondence: Department of Mathematics, Obafemi Awolowo University, Ile-Ife, Nigeria.
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Hence Equation (3) is a five-parameter generalized Burr XII distribution.

Many research work have been done on this distribution. Tadikamalla (1980) studied this distribution and other Burr type distributions. Sarah and Pushkarna (1999) obtained moments of order statistics from doubly truncated Lomax distribution, which is a special case of Burr XII distribution. Begum and Parvin (2002) obtained moments of order statistics from doubly truncated Burr XII distribution while Haseeb and Khan (2002) listed Burr XII among a general class of distributions that satisfies $F(x) = ah(x) + b$, which they characterized by considering conditional moments of function of order statistics.

The Burr XII distribution is an important distribution because it has many other distributions like Pareto II or Lomax distribution (Arnold, 1983; Balakrishnan and Ahsanullah, 1994), log-logistic distribution, compound-Weibull or Weibull-Gamma and Weibull Exponential as particular cases of this distribution. Therefore, expressions for these distributions could be obtained from expressions of Burr XII distribution [Pareto or Lomax at $p = 1$ (Patil and Taillie, 1994), log-logistic at $m = 1, \theta = 1$ (Balakrishnan, Malik and Puthenpura, 1987), Weibull-Gamma at $\theta = 1/\delta$(Tadikamalla, 1980). So, generalizing Burr XII distribution is generalizing all other distributions which come under special cases of Burr XII distribution; hence, in this paper, by defining a suitable random variable, the probability density function of a six-parameter generalized Burr XII distribution is derived. The cumulative distribution function is obtained and its n-th moment is established. The probability density function of some of its order statistics is obtained.

1.   THE SIX-PARAMETER GENERALIZED BURR XII DISTRIBUTION

Theorem 1.   *Suppose Y_1 and Y_2 are independently distributed continuous random variables. If Y_1 has an exponential density function*

$$f(y_1; \theta) = \theta e^{-\theta y_1}, \; y_1 > 0, \; \theta > 0$$

and Y_2 has a gamma distribution with probability density function

$$f(y_2; m, \lambda) = \frac{\lambda^m}{\Gamma m} y_2^{m-1} e^{-\lambda y_2}, \; y_2 > 0, \; m > 0, \; \lambda > 0. \tag{1.1}$$

Then, the random variable

$$X = \sqrt[p]{\frac{Y_1}{Y_2}}$$

has the six-parameter generalized Burr XII distribution with parameters $(\mu = 0, \sigma = 1, \lambda, \theta, m, p)$.

Proof. The joint probability density function of Y_1 and Y_2 is

$$f(y_1, y_2) = \frac{\lambda^m \theta}{\Gamma m} y_2^{m-1} e^{-(\theta y_1 + \lambda Y_2)}, \quad y_1 > 0, \ y_2 > 0, \ m > 0, \ \lambda > 0, \ \theta > 0 \qquad (1.2)$$

Let $x_1 = \sqrt[p]{\dfrac{y_1}{y_2}}$ and $x_2 = y_2$. We obtain the probability density of the random variable X_1 as

$$f(x_1) = \frac{\lambda^m \theta p}{\Gamma m} \int_0^\infty x_1^{p-1} x_2^m e^{-x_2(\theta x_1^p + \lambda)} dx_2 = \frac{\lambda^m m \theta p x_1^{p-1}}{\left(\lambda + \theta x_1^p\right)^{m+1}}, \qquad (1.3)$$

$$x_1 > 0, \ \lambda > 0, \ \theta > 0, \ m > 0, \ p > 0.$$

If we introduce the location parameter μ and the scaled parameter σ in Equation (1.3), we have

$$f(x; \mu, \sigma, \lambda, \theta, m, p) = \frac{\lambda^m m \theta p \left(\frac{x-\mu}{\sigma}\right)^{p-1}}{\sigma \left[\lambda + \theta \left(\frac{x-\mu}{\sigma}\right)^p\right]^{m+1}}, \qquad (1.4)$$

$$x > 0, \ \mu > 0, \ \sigma > 0, \ \lambda > 0, \ \theta > 0, \ m > 0, \ p > 0.$$

This probability density function in Equation (1.4) is what we refer to as a six-parameter generalized Burr XII distribution. For the rest of this paper, we shall assume that $\mu = 0$ and $\sigma = 1$ without loss of generality.

1.1. CUMULATIVE DISTRIBUTION FUNCTION (CDF) OF THE GENERALIZED BURR XII DISTRIBUTION

If a random variable X has the generalized Burr XII probability density function given in the Equation (1.3), the cdf of X is given as

$$F(x) = \lambda^m m \theta p \int_0^x \frac{t^{p-1}}{(\lambda + \theta t^p)^{m+1}} dt = 1 - \lambda^m (\lambda + \theta x^p)^{-m}, \qquad (1.5)$$

$$\lambda > 0, \ \theta > 0, \ m > 0, \ p > 0$$

The probability that a generalized Burr XII random variable X lies in the interval (α_1, α_2) is given as

$$Pr(\alpha_1 < X < \alpha_2) = \lambda^m \left[(\lambda + \theta \alpha_1^p)^{-m} - (\lambda + \theta \alpha_2^p)^{-m} \right], \quad \text{for } \alpha_1 < \alpha_2. \qquad (1.6)$$

Hence, given the value of the parameters λ, θ, p, m and an interval $(\alpha_1 < \alpha_2)$, the probability $Pr(\alpha_1 < X < \alpha_2)$ can be easily computed.

2. MOMENTS OF THE GENERALIZED BURR XII DISTRIBUTION

The n-th moment of a random variable X that has the generalized Burr XII distribution is

$$E[X^n] = \int_{-\infty}^{\infty} x^n f(x)\,dx \tag{2.1}$$

$$= \lambda^m m\theta p \int_0^{\infty} \frac{x^{n+p-1}}{(\lambda + \theta x^p)^{m+1}}\,dx = \frac{\theta m p}{\lambda} \int_0^{\infty} \frac{x^{n+p-1}}{\left(1 + \frac{\theta}{\lambda} x^p\right)^{m+1}}\,dx \tag{2.2}$$

Let $z = \theta x^p/\lambda$, then $x = \sqrt[p]{\frac{\lambda}{\theta}z}$. So

$$E[X^n] = \frac{\lambda^{n/p} m}{\theta^{n/p}} \int_0^{\infty} \frac{z^{n/p}}{(1+z)^{m+1}}\,dz = \frac{\lambda^{n/p} m}{\theta^{n/p}} B(n/p + 1, m - n/p) \tag{2.3}$$

$$= m\left(\frac{\lambda}{\theta}\right)^{n/p} \frac{\Gamma(n/p + 1)\Gamma(m - n/p)}{\Gamma(m+1)} = \left(\frac{\lambda}{\theta}\right)^{n/p} \frac{\Gamma(n/p + 1)\Gamma(m - n/p)}{\Gamma m} \quad m > n/p. \tag{2.4}$$

When $n = 1$ in Equation (3.4), we obtain the mean of the generalized Burr XII distribution as

$$E[X] = \left(\frac{\lambda}{\theta}\right)^{1/p} \frac{\Gamma(1/p + 1)\Gamma(m - 1/p)}{\Gamma m}, \quad m > 1/p. \tag{2.5}$$

Also, when $n = 2$,

$$E[X^2] = \left(\frac{\lambda}{\theta}\right)^{2/p} \frac{\Gamma(2/p + 1)\Gamma(m - 2/p)}{\Gamma m}, \quad m > 2/p. \tag{2.6}$$

Hence, the variance of the generalized Burr XII distribution is obtained as

$$\sigma_X^2 = \left(\frac{\lambda}{\theta}\right)^{2/p} \left[\frac{\Gamma(2/p + 1)\Gamma(m - 2/p)}{\Gamma m} - \left(\frac{\Gamma(1/p + 1)\Gamma(m - 1/p)}{\Gamma m}\right)^2\right] \tag{2.7}$$

3. ORDER STATISTICS OF THE GENERALIZED BURR XII DISTRIBUTION

Let $X_1, X_2. \dots, X_n$ be n independently continuous random variables from the generalized Burr XII distribution and let $X_{1:n} \le X_{2:n} \le \cdots \le X_{n:n}$ be the corresponding order statistics. Let $F_{X_{(r:n)}}(x)$, $(r = 1, 2, \dots, n)$ be the cumulative distribution function of the r-th order statistics $X_{(r:n)}$ and $f_{X_{(r:n)}}(x)$ denotes its probability density function. David (1970) gives the probability density function of $X_{(r:n)}$ as

$$f_{X_{(r:n)}}(x) = \frac{1}{B(r, n-r+1)} P^{r-1}(x)[1 - P(x)]^{n-r} p(x). \tag{3.1}$$

For the generalized Burr XII distribution with probability density function and cumulative distribution function given in the Equations (1.3) and (1.5) respectively, by substituting $f(x)$ for $p(x)$ and $F(x)$ for $P(x)$ in the Equation (3.1), we have

$$f_{X_{(r:n)}}(x) = \frac{1}{B(r, n-r+1)} \left[1 - \frac{\lambda^m}{(\lambda + \theta x^p)^m} \right]^{r-1} \left[\frac{\lambda^m}{(\lambda + \theta x^p)^m} \right]^{n-r} \frac{\lambda^m \theta p x^{p-1}}{(\lambda + \theta x^p)^{m+1}}$$

$$= \frac{\lambda^{m(n-r+1)} m \theta p}{B(r, n-r+1)} \frac{x^{p-1}[(\lambda + \theta x^p)^m - \lambda^m]^{r-1}}{(\lambda + \theta x^p)^{mn+1}} \tag{3.2}$$

3.1. THE PROBABILITY DENSITY FUNCTION OF THE MINIMUM AND MAXIMUM OBSERVATIONS FROM THE GENERALIZED BURR XII DISTRIBUTION

The minimum observation is denoted as $X_{1:n}$ and its probability density function is obtained by making $r = 1$ in the Equation (3.2) to have

$$f_{X_{(1:n)}}(x) = \frac{\lambda^{mn} mn \theta p x^{p-1}}{(\lambda + \theta x^p)^{mn+1}}, \quad x > 0, \ \lambda > 0, \ \theta > 0, \ m > 0; \tag{3.3}$$

which is another generalized Burr XII distribution with parameter $(\lambda, \theta, mn; p)$.

The maximum observation is denoted by $X_{n:n}$ and its probability density function is obtained by making $r = n$ in the Equation (3.2) to have

$$f_{X_{(n:n)}}(x) = \frac{\lambda^m mn \theta p x^{p-1}\{(\lambda + \theta x^p)^m - \lambda^m\}^{n-1}}{(\lambda + \theta x^p)^{mn+1}}, x > 0, \lambda > 0, \theta > 0, p > 0, m > 0; \tag{3.4}$$

The q-th moment of the minimum observation from the generalized Burr XII distribution is

$$E\left[X_{(1:n)}^{q}\right] = \int x^{q} f_{X_{(1:n)}}(x)\, dx$$

$$= \lambda^{mn} mn\theta p \int_{0}^{\infty} \frac{x^{p+q-1}}{(\lambda + \theta x^{p})^{mn+1}}\, dx = mn \left(\frac{\lambda}{\theta}\right)^{q/p} B\left(\frac{q}{p}+1, mn - \frac{q}{p}\right), \tag{3.5}$$

where $B(.,.)$ is a complete beta function.

Hence, the mean of the minimum observation from the generalized Burr XII distribution is

$$E\left[X_{(1:n)}\right] = mn \left(\frac{\lambda}{\theta}\right)^{1/p} B\left(\frac{1}{p}+1, mn - \frac{1}{p}\right), \tag{3.6}$$

while the variance is

$$Var\left[X_{(1:n)}\right] = mn \left(\frac{\lambda}{\theta}\right)^{2/p} \left[B\left(\frac{2}{p}+1, mn - \frac{2}{p}\right) - mn \left(B\left(\frac{1}{p}+1, mn - \frac{1}{p}\right)\right)^{2} \right]. \tag{3.7}$$

4. A RELATIONSHIP BETWEEN THE GENERALIZED BURR XII AND OTHER STATISTICAL DISTRIBUTION

In this section, we shall state and prove a theorem that relates the generalized Burr XII distribution to another statistical distribution.

Theorem 2.   *Suppose Y is a continuously distributed random variable with probability density function $f_Y(y)$, then the random variable*

$$X = \sqrt[p]{e^{\frac{Y}{m}} - \frac{\lambda}{\theta}}$$

has a generalized Burr XII distribution with parameters (p, m, θ, λ) if Y is exponentially distributed.

Proof: The probability density function of an exponential random variable Y is

$$f_Y(y) = e^{-y}, \quad y > 0.$$

By omitting all constants, the density of X can be written as

$$f_X(x) \propto \frac{x^{p-1}}{(\lambda + \theta x^p)^{mn+1}} \tag{4.1}$$

Since any density function proportional to the right hand side of the Equation (4.1) is that of a generalized Burr XII random variable, the proof is complete.

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