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GLOBALLY STABLE PERIODIC SOLUTION OF CERTAIN FOURTH ORDER NON-LINEAR DIFFERENTIAL EQUATIONS

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Abstract

In this paper, we give criteria for the existence of a unique solution to a certain fourth order nonlinear differential equations which is bounded together with its derivatives on the real line, globally stable and periodic by the use of a complete Lyapunov function.

1. Introduction

In this paper, we study the fourth order nonlinear differential equation

$$x^{(iv)} + a\dddot{x} + f(x, \dot{x})\ddot{x} + g(\dot{x}) + h(x) = p(t), \tag{1.1}$$

where a is a positive constant, the functions f, g, h and p are continuous in the respective argument displayed explicitly. The studies of the qualitative properties

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(boundedness, stability and periodicity) of solutions for higher order nonlinear differential equations have been a subject of interest that have received considerable attention from several scholars who have obtained interesting results. Some of these results have been summarized in [14].

In [1], the authors employed the frequency domain method to investigate the boundedness of this class of equation.

In [11], the Cauchy formula for the particular solution of non-homogeneous linear differential equation was employed to achieve the results on boundedness of solution.

Other articles in this connection include Tiryaki and Tunc [18], Tunc [19-22], and Tunc and Tiryaki [23] where the second method of Lyapunov was used. All these results in one way or the other generalize some results on third order nonlinear equations (see [2, 5, 12 and 16]).

In [22], the author gave criteria for the asymptotic stability and boundedness of solutions of certain class of the equation above by the use of an incomplete Lyapunov (Yoshizawa [24]) function and a stringent condition was placed on the nonlinear terms g and h which is the necessity for these functions not only to be continuous but also be differentiable.

In [3], the authors developed a theory to discuss these qualitative properties (boundedness, stability and periodicity) in unified way using the Lyapunov second method. This theory was then adapted for certain equations of third order in [12, 13].

As in [11], we will consider the equation (1.1) with an equivalent system

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= z, \\ \dot{z} &= w, \\ \dot{w} &= -aw - f(x, y)z - g(y) - h(x) + p(t), \end{aligned} \tag{1.2}$$

this time with the focus on the boundedness, stability and periodicity properties of solution in a unified way.

Since the second (direct) method of Lyapunov still remains one of the most effective methods to study these concepts, the purpose of this paper is to extend the study in [12, 13] to certain equations of fourth order and give sufficient criteria on the nonlinear terms f, g and h that will guarantee the existence of a unique solution to the equation (1.1) which is bounded together with its derivatives on a real line, globally stable and periodic. This we shall achieve by the use of a single complete Lyapunov function without the use of any signum function and less restriction on the nonlinear terms g and h other than been continuous.

Even though there is no unique way of constructing a Lyapunov function, we adapted Cartwright [4] for the construction of the Lyapunov function used in this work.

We wish to refer the reader to [4], [10], [15], [16], [17], [19], [20] and [24] for terminologies, standard results and techniques.

The paper is organized in the following order: Section 2 gives definitions and theories behind our result. Our main result features in Section 3, preliminary results in proving the main result are given in Section 4. Section 5 features the proof of the main result of this paper.

Notation. Throughout this paper $K, K_0, K_1, \ldots, K_{12}$ will denote finite positive constants. $K_i's$ are not necessarily the same for each time they occur, but each K_i, $i = 1, 2, \ldots$ retains its identity throughout.

By $V|_{1.2}$ we mean the function V constructed along the solution part of the system (1.2).

2. Generalized Theorems

In order to reach our main results, we will first give some important basic definitions for the general non-autonomous differential system.

We consider the system

$$X = F(t, X), \tag{2.1}$$

where $F \in C[I \times S_\rho]$, $I = [0, \infty)$, $t > 0$ and $S_\rho = \{X \in \Re^n : |X| < \rho\}$. Assume that F is smooth enough to ensure the existence and uniqueness of solutions of (2.1) through every point $(t_0, x_0) \in J \times S_\rho$. Also, let $F(t, 0) \equiv 0$ so that (2.1) admits the zero solution $X \equiv 0$.

Definition 2.1 [24]. The solution $X(t) \equiv 0$ of (2.1) is stable if for any $\varepsilon > 0$ and any $t_0 \in I$ there exists a $\delta(t_0, \varepsilon) < 0$ such that if $X_0 \in S_{\delta(t_0, \varepsilon)}$, then $X(t; t_0, X_0) \in S_\varepsilon$ for all $t \geq t_0$.

Definition 2.2 [24]. The solution $X(t) \equiv 0$ of (2.1) is asymptotically stable in the whole (globally asymptotically stable) if it is stable and every solution of (1.1) tends to zero as $t \to \infty$.

Definition 2.3 [24]. The solution $X(t) \equiv 0$ of (2.1) is uniformly asymptotically stable if it is stable and there exists a $\delta(t_0) > 0$ such that $\| X(t; t_0, X_0) \| \to 0$ as $t \to \infty$ for all $X_0 \in S_{\delta_0}$.

Definition 2.4 [24]. A solution $X(t)$ of (2.1) is said to be *bounded* if there exists a $\beta > 0$, such that

$$| X(t; t_0, x_0) | < \beta$$

for all $t \geq t_0$, where β may depend on each solution.

Definition 2.5 [24]. A solution $X(t)$ of (2.1) is said to be *equi-bounded*, if for any α and $t_0 > I$, there exists a $\beta(t_0, \alpha) > 0$, such that if $X_0 \in S_\alpha$, then

$$| X(t; t_0, x_0) | < \beta(t_0, \alpha)$$

for all $t \geq t_0$, where α is the length of the interval, i.e., $\alpha \in [t_1, t_2]$, $t_0 \leq t_1 \leq t_2 \leq t$.

Definition 2.6 [24]. A solution $X(t)$ of (2.1) is said to be *uniformly-bounded*, if for any α and $t_0 > I$, there exists a $\beta(\alpha) > 0$, such that if $X_0 \in S_\alpha$, then

$$| X(t; t_0, x_0) | < \beta(\alpha)$$

for all $t \geq t_0$, where α is as defined above

Definition 2.7 [24]. A solution $X(t)$ of (2.1) is said to be *ultimately-bounded* for bound $\mathbf{M}$, if there exist $\mathbf{M} > 0$ and $\mathbf{T} > 0$, such that for every solution $X(t; t_0, X_0)$ of (2.1)

$$| X(t; t_0, x_0) | < \mathbf{M}$$

for all $t \geq t_0 + \mathbf{T}$.

Definition 2.8 [24]. A solution $X(t)$ of (2.1) is said to be *uniformly ultimately-bounded* for bound $\mathbf{M}$, if there exists $\mathbf{M} \geq 0$ and if corresponding to any $\alpha > 0$ and

$t_0 \in I$ there exists $\mathbf{T}(\alpha) > 0$, such that $X_0 \in S_\alpha$ implies that

$$| X(t; t_0, x_0) | < \mathbf{M}$$

for all $t \geq t_0 + \mathbf{T}(\alpha)$.

Definition 2.9 [24]. A real value function V defined as $V : I \times \Re^n \to \Re$, of a real variable X $(X \in \Re^n)$ and t with the conditions that $t \geq T$ and $| x_i | < H$ (T and H are real constants of which T can be supposed to be as large as we wish and H as small as we wish but not zero) having the properties:

(i) Continuity: $V(t, X)$ is continuous and single valued under the condition stated above and $V(t, 0) \equiv 0$.

(ii) $V(t, X)$ is positive definite and

(iii) $\dot{V} = \frac{\partial V}{\partial x_1}\dot{x}_1 + \frac{\partial V}{\partial x_2}\dot{x}_2 + \cdots + \frac{\partial V}{\partial x_n}\dot{x}_n$, representing the total derivative with respect to t is negative definite, is called a *Lyapunov function*.

We shall give the following definitions in our own context:

Definition 2.10. A Lyapunov function V defined as $V : I \times \Re^n \to \Re$ is said to be *complete* if for $X \in \Re^n$,

(i) $V(t, X) \geq 0$

(ii) $V(t, X) = 0$, if and only if $X = 0$ and

(iii) $\dot{V}|_{2.1}(t, X) \leq -c| X |$, where c is any positive constant and $| X |$ is given by $| X | = \left(\sum_{i=1}^{n} (x_i^2) \right)^{1/2}$ such that $| X | \to \infty$ as $X \to \infty$.

Definition 2.11. A Lyapunov function V defined as $V : I \times \Re^n \to \Re$ is said to be *incomplete* if for $X \in \Re^n$, conditions (i) and (ii) of Definition 2.10 are satisfied, and in addition (iii) $\dot{V}(t, X)|_{2.3} \leq -c| X |_*$, where c is any positive constant and $| X |_*$ is given as $| X |_* = \left(\sum_{i=1}^{<n} x^2 \right)^{1/2}$ such that $| X |_* \to \infty$ as $X \to \infty$.

To make these definitions of complete and incomplete Lyapunov functions clearer we shall consider a simple case of $n = 2$.

Consider the simple 2nd order linear differential equation

$$\ddot{x} + a\dot{x} + bx = 0,$$

(where a and b are all positive) with an equivalent system

$$\dot{x} = y,$$

$$\dot{y} = -ay - bx. \tag{2.2}$$

The following are some of the possible Lyapunov functions for the system:

$$2V = \left(\frac{c+\delta}{a}\right)bx^2 + \left(\frac{c+\delta}{a}\right)y^2, \tag{2.3}$$

$$2V = \left(\frac{cb^2 + \delta a^2}{ab}\right)x^2 + \left(\frac{c}{a}\right)y^2 + 2\frac{\delta}{b}xy \tag{2.4}$$

and

$$2V = \left(\frac{b^2(b+c) + \delta a^2}{ab}\right)x^2 + \left(\frac{\delta + c}{a}\right)y^2 + 2\frac{\delta}{b}xy, \tag{2.5}$$

where c and δ are positive constants.

Let $(x(t), y(t))$ be any solution of (2.2). Then by a straightforward calculation from (2.3)-(2.5) and (2.2), we observe that

$$\dot{V} = -\delta y^2,$$

$$\dot{V} = -\delta x^2$$

and

$$\dot{V} = -\delta(x^2 + y^2)$$

are the derivatives of V with respect to the system (2.2), respectively.

Lyapunov functions defined as in (2.3) and (2.4) are referred to as incomplete while the one defined by (2.5) is complete.

We give the following standard theorems on Lyapunov functions relevant for this work.

In an attempt to discuss the unified theory of periodicity of dissipative ordinary differential equations, Burton and Shunian [3] considered the general differential equation

$$\dot{X} = F(t, X). \tag{2.6}$$

When the equation (2.6) is linear, it is written as

$$X = A(t)X + P(t), \tag{2.7}$$

with the homogeneous system

$$X = A(t)X, \tag{2.8}$$

where $A(t)$ is an $n \times n$ matrix.

The use of Lyapunov functions which led to the formulation of the following scheme was employed:

(i) If $F(t, 0) = 0$, and if there is a function $V : [0, \infty) \times \Re^3 \to \Re$ such that

$$W_1(|X|) \le V(t, X) \le W_2(|X|)$$

and

$$V(t, X)|_{(2.1)} \le -W_3(|X|),$$

where W_i $(i = 1, 2, 3)$ are strictly increasing continuous function defined as $W_i : [0, \infty) \to [0, \infty)$ with $W(s) > 0$ and $W(0) = 0$ as wedges. Then the solutions of the equation (2.6) are uniformly asymptotically stable (UAS).

(ii) If there is a function $V : [0, \infty) \times \Re^3 \to \Re$ such that

$$W_1(|X|) \le V(t, X) \le W_2(|X|)$$

and

$$V(t, X)|_{(2.1)} \le -W_3(|X|) + M \ (M > 0),$$

then the solutions of the equation (2.6) are ultimately bounded (UB) and uniformly ultimately bounded (UUB).

(iii) If the solutions of the equation (2.3) and the equation (2.7) are unique, UB and UUB, then the equation (2.6) has a periodic solution.

(iv) If the zero solutions of the equation (2.5) are uniformly asymptotically stable (UAS), then the equation (2.7) has a globally stable periodic solution.

We shall now state without proof, Theorems of Burton and Shunian [3].

Theorem A [3]. *If F is Lipschitz in X and periodic in t with period T and if the solutions are uniformly bounded and uniformly ultimately bounded for any given bound* (*say*) *B, then the equation* (2.6) *has a T-periodic solution.*

Theorem B [3]. *Let the following conditions hold*:

(a) $F(t+T, X) = F(t, X)$ *for all t and some $T > 0$*,

(b) *all solutions of the equation* (2.6) *are bounded*,

(c) *each solution of the equation* (2.6) *is equi-asymptotically stable*,

(d) *the zero solution of the homogeneous system corresponding to the equation* (2.6) *is uniformly asymptotically stable* (*UAS*).

Then the equation (2.6) *has a globally stable T-periodic solution*

3. Main Results

We give the main result of this work:

Theorem 3.1. *Let f, g, h and p be continuous and also in addition let p and h be periodic with period ω, and the following conditions hold*:

(i) $H_0 = \dfrac{h(x) - h(0)}{x} \leq d \in I_0,\ \ x \neq 0$ *and* $G_o = \dfrac{g(y) - g(0)}{y} \leq c;\ \ y \neq 0$,

(ii) $h(0) = g(0) = 0$,

(iii) $|f(x, y)| \leq b$,

(iv) $|p(t)| \leq M$ (*constant*) *for all* $t \geq 0$.

Then (1.1) *has a globally stable ω-periodic solution.*

4. Preliminary Results

We shall use as a tool to prove our main results besides the equation (1.1) a function $V(x, y, z, w)$ defined by

$$2V(x, y, z, w) = Ax^2 + By^2 + Cz^2 + Dw^2 + 2Exy$$
$$+ 2Fxz + 2Gxw + 2Hyz + 2Iyw + 2Jzw, \tag{4.1}$$

where

$$A = \frac{a\delta}{\Delta}\{(b+d)(c^2+d^2)[d(1-ad)-c]+d^3[a(b^2+d^2)+L]\},$$

$$B = \frac{\delta}{\Delta}\{dL(abd+c)+a(b^2+d^2)[b(d-c)+cd]$$

$$+[d(1-ad)-c][ad(b^2+c^2)-cd^2(b+1)+a^2bc]\},$$

$$C = \frac{\delta}{\Delta}\{a(b^2+d^2)[d(1-ad+a^2c+d)-c]$$

$$+d[c(a^2+b^2)-ab][d(1-ad)-c]+dL(a^2c+d)\},$$

$$D = \frac{cd\delta}{\Delta}\{L+ab^2+(d-c)+ab[(1-ad)-c]\},$$

$$E = \frac{ac\delta}{\Delta}\{d^2L+(b^2+d^2)(d-c)\},$$

$$F = \frac{cd\delta}{bd\Delta}\{d^2L+ad^2(b^2+d^2)+[b(a^2+d^2)+d^2][ab^2d^2[d(1-ad)-c]]\},$$

$$G = \frac{abc[d(1-ad-c)]\delta}{\Delta},$$

$$H = \frac{abcd\delta}{\Delta}\{a(b^2+d^2)+L\},$$

$$I = \frac{a\delta}{\Delta}\{d^2L+bd[d(1-ad)-c]+(b^2+d^2)(d-c)\},$$

$$J = \frac{acd\delta}{\Delta}\{ab^2+d-c+L\},$$

$$\Delta = abcd[d(1-ad)-c],$$

$$L = b[ad+c[c(b+1)-c]] \tag{4.2}$$

with a, b, c, d positive and $[d(1-ad)-c] > 0$.

Lemma 4.1. *Subject to the assumptions of Theorem* 3.1 *there exist positive constants* $K_i = K_i(a, b, c, d, \delta)$, $i = 1, 2$ *such that*

$$K_1(x^2+y^2+z^2+w^2) \le V(x, y, z, w) \le K_2(x^2+y^2+z^2+w^2). \tag{4.3}$$

Proof. Clearly $V(0, 0, 0, 0) \equiv 0$.

By rearranging (4.1), we have

$$2V(x, y, z, w)$$

$$= \left(\frac{\delta}{\Delta}\right)\left\{a[d(1-ad)]\left\{b(cx+dy+w)^2 + d^2(y+b^3d^2x)^2\right.\right.$$

$$\left. + b^2d(y+a^2bdx)^2 + acd\left(z + \frac{b^2d^3}{a}x\right)^2\right\}$$

$$+ dL\left\{(z+acx)^2 + ac^2\left(z+\frac{1}{a}w\right)^2 + c\left(y+\frac{ad}{c}w\right)^2 + ad^2\left(x+\frac{c}{d}y\right)^2\right.$$

$$\left. + abd\left(y+\frac{c}{d}z\right)^2\right\} + ad(b^2+d^2)\left\{ad^2\left(x+\frac{c(d-c)}{ad^3}y\right)^2\right.$$

$$+ a^2c\left(z+\frac{d}{a}x\right)^2 + \frac{c}{a(b^2+d^2)}(w+az)^2 + b(d-c)\left(y+\frac{w}{b}\right)^2$$

$$\left. + c(y+abz)^2\right\} + \left\{[d(1-ad)-c](ad(c^2+d^2)+abd^2) - \frac{cd^3}{a}(b^2+d^2)\right.$$

$$\left. - b^4cd^3 - a^5b^4d^3 - ab^6d^4 - a^2c^2d^2L\right\}x^2$$

$$+ \left\{[d(1-ad)-c][ad(b^2+c^2) - cd^2(b+1) + a^2bc - abd^2]\right.$$

$$\left. - ac^2dL - \frac{c^2(d-c)^2}{d^3}\right\}y^2 + \{ad^2(b^2+d^2) + d(b^2c-ab)[d(1-ad)-c]$$

$$- a^3b^2cd(b^2+d^2) - abc^2L - a^2cd[ab^2+(d-c)]\}z^2$$

$$\left. + \left\{L - ab[d(1-ad)-c] - \frac{a}{b}(b^2+d^2)(d-c) - \frac{a^2d^3}{c} - cdL\right\}w^2\right\}. \tag{4.4}$$

From which we obtain

$$
\begin{aligned}
&2V(x, y, z, w)\\
&\geq \left(\frac{\delta}{\Delta}\right)\Bigg\{\bigg\{[d(1-ad)-c](ad(c^2+d^2)+abd^2)\\
&-\frac{cd^3}{a}(b^2+d^2)-b^4cd^3-a^5b^4d^3-ab^6d^4-a^2c^2d^2L\bigg\}x^2\\
&+\bigg\{[d(1-ad)-c][ad(b^2+c^2)-cd^2(b+1)+a^2bc-abd^2]\\
&-ac^2dL-\frac{c^2(d-c)^2}{d^3}\bigg\}y^2+\{ad^2(b^2+d^2)+d(b^2c-ab)[d(1-ad)-c]\\
&-a^3b^2cd(b^2+d^2)-abc^2L-a^2cd[ab^2+(d-c)]\}z^2\\
&+\bigg\{L-ab[d(1-ad)-c]-\frac{a}{b}(b^2+d^2)(d-c)-\frac{a^2d^3}{c}-cdL\bigg\}w^2\Bigg\}\\
&\geq K_1(x^2+y^2+z^2+w^2),
\end{aligned}
\tag{4.5}
$$

where

$$
\begin{aligned}
K_1=\frac{\delta}{\Delta}\cdot\min\Bigg\{&\bigg|[d(1-ad)-c](ad(c^2+d^2)+abd^2)\\
&-\frac{cd^3}{a}(b^2+d^2)-b^4cd^3-a^5b^4d^3-ab^6d^4-a^2c^2d^2L\bigg|,\\
&\bigg|[d(1-ad)-c][ad(b^2+c^2)-cd^2(b+1)+a^2bc-abd^2]\\
&-ac^2dL-\frac{c^2(d-c)^2}{d^3}\bigg|,\ \bigg|ad^2(b^2+d^2)+d(b^2c-ab)[d(1-ad)-c]\\
&-a^3b^2cd(b^2+d^2)-abc^2L-a^2cd[ab^2+(d-c)]\bigg|,\\
&\bigg|L-ab[d(1-ad)-c]-\frac{a}{b}(b^2+d^2)(d-c)-\frac{a^2d^3}{c}-cdL\bigg|\Bigg\}.
\end{aligned}
$$

Therefore

$$2V(x, y, z, w) \geq K_1(x^2 + y^2 + z^2 + w^2). \tag{4.6}$$

By using the Schwarz inequality $|xy| \leq \frac{1}{2}|x^2 + y^2|$, on the equation (4.1), we have

$$\begin{aligned} 2V(x, y, z, w) &\leq \left(\frac{\delta}{\Delta}\right)\{[A + E + F + G]x^2 + [B + E + H + I]y^2 \\ &\quad + [C + F + H + J]z^2 + [D + G + I + J]w^2\} \\ &\leq K_2(x^2 + y^2 + z^2 + w^2), \end{aligned} \tag{4.7}$$

where

$$\begin{aligned} K_2 = \left(\frac{\delta}{\Delta}\right)\max\{&[A + E + F + G], [B + E + H + I], \\ &[C + F + H + J], [D + G + I + J]\} > 0 \end{aligned}$$

From inequalities (4.6) and (4.7), we have

$$K_1(x^2 + y^2 + z^2 + w^2) \leq V(x, y, z, w) \leq K_2(x^2 + y^2 + z^2 + w^2). \tag{4.8}$$

This proves Lemma 4.1.

Lemma 4.2. *Subject to the assumptions of Theorem* 3.1, *there exist positive constants* $K_j = K_j(a, b, c, d, \delta)$ $(j = 3, 4)$ *such that for any solution* (x, y, z, w) *of system* (1.2),

$$\begin{aligned} \dot{V}|_{(1.2)} &\equiv \frac{d}{dt}V|_{(1.2)}(x, y, z, w) \\ &\leq -K_3(x^2 + y^2 + z^2 + w^2) + K_4(|x| + |y| + |z| + |w|)|p(t)|. \end{aligned} \tag{4.9}$$

Proof. From equations (1.1) and (1.2), we have

$$\begin{aligned} \dot{V}|_{(1.2)} &= \frac{\partial V}{\partial x}\dot{x} + \frac{\partial V}{\partial y}\dot{y} + \frac{\partial V}{\partial z}\dot{z} + \frac{\partial V}{\partial w}\dot{w}. \\ &= \frac{\partial V}{\partial x}y + \frac{\partial V}{\partial y}z + \frac{\partial V}{\partial z}w + \frac{\partial V}{\partial z}(-aw - bz - g(y) - h(x) + p(t)). \end{aligned} \tag{4.10}$$

After simplification, we have

$$\dot{V} = \left(\frac{\delta}{\Delta}\right)\{-Gh(x)x - Ig(y)y - [Jb - H]z^2 - [Da - J]w^2$$

$$- Gg(y)x - Ih(x)y - [Gb - E]xz - Jh(x)z$$

$$-[Ga - F]xw - Dh(x)w - [Ib - F - B]yz - Jg(y)z$$

$$-[Ia - G - H]yw - Dg(y)w - [Db + Ja - I - C]zw$$

$$+ Ey^2 + Axy + p(t)[Gx + Iy + Jz + Dw]\}. \tag{4.11}$$

Using the conditions on $h(x)$ and $g(y)$ the equation (4.11) becomes

$$\dot{V} \le \left(\frac{\delta}{\Delta}\right)\{-Gdx^2 - [Ic - E]y^2 - [Jb - H]z^2 - [Da - J]w^2$$

$$-[Gc + Id - A]xy - [Gb + Jd - E]xz - [Ga + Dd - F]xw$$

$$-[Ib + Jc - F - B]yz - [Ia + Dc - G - H]yw$$

$$-[Db + Ja - I - C]zw[h(0) + g(0) + p(t)][Gx + Iy + Jz + Dw]\}. \tag{4.12}$$

Which reduces to

$$\dot{V} \le \left(\frac{\delta}{\Delta}\right)\{-K_3(x^2 + y^2 + z^2 + w^2) + [h(0) + g(0) + p(t)][Gx + Iy + Jz + Dw]\}, \tag{4.13}$$

where $K_3 = \max\{Gd, [Ic - E], [Jb - H], [Da - J]\}$.

Inequality (4.13) further reduces to

$$\dot{V} \le \left(\frac{\delta}{\Delta}\right)\{-K_3(x^2 + y^2 + z^2 + w^2) + K_4(|x| + |y| + |z| + |w|)p(t)\} \tag{4.14}$$

with $K_4 = \max\{D, G, I, J\}$.

Therefore

$$\dot{V} \le -K_5(x^2 + y^2 + z^2 + w^2) + K_6(|x| + |y| + |z| + |w|)p(t), \tag{4.15}$$

where $K_5 = \left(\frac{\delta}{\Delta}\right)K_3$ and $K_6 = \left(\frac{\delta}{\Delta}\right)K_4$.

Since

$$(|x| + |y| + |z| + |w|) \leq 2(x^2 + y^2 + z^2 + w^2)^{1/2},$$

inequality (4.15) becomes

$$\frac{dV}{dt} \leq -K_5(x^2 + y^2 + z^2 + w^2) + K_7(x^2 + y^2 + z^2 + w^2)^{1/2}|p(t)|, \quad (4.16)$$

where $K_7 = 2K_6$.

This completes the proof of Lemma 4.2.

5. Proof of the Main Results

We shall now give the proof of the main result.

Proof of Theorem 3.1. From Lemma 4.1 and Lemma 4.2, we established condition (d) of Theorem B. By the hypothesis of Theorem 3.1, condition (a) of Theorem B is also satisfied.

We need now to show that under the same conditions of Theorem 3.1, condition (b) of Theorem B is also satisfied.

Indeed from the inequality (4.16),

$$\frac{dV}{dt} \leq -K_5(x^2 + y^2 + z^2 + w^2) + K_7(x^2 + y^2 + z^2 + w^2)^{1/2}|p(t)|,$$

and also from the inequality (4.6), we have

$$(x^2 + y^2 + z^2 + w^2)^{1/2} \leq \left(\frac{2V}{K_1}\right)^{1/2}$$

Thus, the inequality (4.16) becomes

$$\frac{dV}{dt} \leq -K_8 V + K_9 V^{1/2}|p(t)|. \quad (5.1)$$

We note that

$$K_5(x^2 + y^2 + z^2 + w^2) = K_5 \cdot \frac{V}{K_1}$$

and

$$\frac{dV}{dt} \le -K_8 V + K_9 V^{1/2} |\, p(t) \,|, \tag{5.2}$$

where $K_8 = \dfrac{K_6}{K_2}$ and $K_9 = \dfrac{K_7}{K_2^{1/2}}$

These imply that

$$\dot{V} \le -K_8 V + K_9 V^{1/2} |\, p(t) \,|$$

and this can be written as

$$\dot{V} \le -2K_{10} V + K_9 V^{1/2} |\, p(t) \,|, \tag{5.3}$$

where $K_{10} = \dfrac{1}{2} K_8$

Therefore

$$\dot{V} + K_{10} V \le -K_{10} V + K_9 V^{1/2} |\, p(t) \,| \tag{5.4}$$

$$\le K_9 V^{1/2} \{|\, p(t) \,| - K_{11} V^{1/2}\}, \tag{5.5}$$

where $K_{11} = \dfrac{K_{10}}{K_9}$

Thus, the inequality (5.5) becomes

$$\dot{V} + K_{10} V \le K_9 V^{1/2} V^*, \tag{5.6}$$

where

$$V^* = |\, p(t) \,| - K_{11} V^{1/2} \tag{5.7}$$

$$\le V^{1/2} |\, p(t) \,|$$

$$\le |\, p(t) \,|. \tag{5.8}$$

When $|\, p(t) \,| \le K_{11} V^{1/2}$,

$$V^* \le 0 \tag{5.9}$$

and when $|\,p(t)\,| \geq K_{11}V^{1/2}$,

$$V^* \leq |\,p(t)\,| \cdot \frac{1}{K_{11}}. \tag{5.10}$$

On substituting the inequality (5.9) into the inequality (5.5), we have

$$\dot{V} + K_{10}V \leq K_{12}V^{1/2}|\,p(t)\,|,$$

where

$$K_{12} = \frac{K_9}{K_{11}}$$

This implies that

$$V^{-1/2}\dot{V} + K_{10}V^{1/2} \leq K_{12}|\,p(t)\,|. \tag{5.11}$$

Multiplying both sides of the inequality (4.11) by $e^{1/2K_{10}t}$, we have

$$e^{1/2K_{10}t}\{\,V^{-1/2}\dot{V} + K_{10}V^{1/2}\} \leq e^{1/2K_{10}t}K_{12}|\,p(t)\,|, \tag{5.12}$$

i.e.,

$$2\frac{d}{dt}\{V^{1/2}e^{1/2K_{10}t}\} \leq e^{1/2K_{10}t}K_{12}|\,p(t)\,|. \tag{5.13}$$

Integrating both sides of (5.13) from t_0 to t, gives

$$\{V^{1/2}e^{1/2K_{10}\gamma}\}_{t_0}^{t} \leq \int_{t_0}^{t}\frac{1}{2}e^{1/2K_9\tau}K_{12}|\,p(\tau)d\tau\,| \tag{5.14}$$

which implies that

$$\{V^{1/2}(t)\}e^{1/2K_{10}t} \leq V^{1/2}(t_0)e^{1/2K_{10}t_0} + \frac{1}{2}K_{12}\int_{t_0}^{t}|\,2(\tau)\,|e^{1/2K_{10}\tau}d\tau$$

or

$$V^{1/2}(t) \leq e^{-1/2K_{10}t}\left\{V^{1/2}(t_0)e^{1/2K_{10}t_0} + \frac{1}{2}K_{12}\int_{t_0}^{t}|\,p(\tau)\,|e^{1/2K_{10}\tau}d\tau\right\}.$$

Using (4.5) and (4.6), we have

$$K_1(x^2(t)+\dot{x}^2(t)+\ddot{x}^2(t)+\dddot{x}(t))$$

$$\leq e^{-1/2K_{10}t}\left\{K_2(x^2(t_0)+\dot{x}^2(t_0)+\ddot{x}^2(t_0)+\dddot{x}(t_0))e^{1/2K_{10}t_0}\right.$$

$$\left.+\frac{1}{2}K_{12}\int_{t_0}^{t}|p(\tau)|e^{1/2K_{10}\tau}d\tau\right\}^2 \tag{5.15}$$

for all $t \geq t_0$.

Thus

$$x^2(t)+\dot{x}^2(t)+\ddot{x}^2(t)+\dddot{x}(t)$$

$$\leq \frac{1}{K_1}\left\{e^{-\frac{1}{2}K_{10}t}\left\{K_2(x^2(t_0)+\dot{x}^2(t_0)+\ddot{x}^2(t_0)+\dddot{x}(t_0))e^{\frac{1}{2}K_{10}t_0}\right.\right.$$

$$\left.\left.+\frac{1}{2}K_{12}\int_{t_0}^{t}|p(\tau)|e^{\frac{1}{2}K_{10}\tau}d\tau\right\}^2\right\}$$

$$\leq \left\{e^{-\frac{1}{2}K_{10}t}\left\{A_1+A_2\int_{t_0}^{t}|p(\tau)|e^{\frac{1}{2}K_{10}\tau}d\tau\right\}^2\right\}, \tag{5.16}$$

where A_1 and A_2 are constants depending on $\{K_1, K_2, ..., K_{12}$ and $(x^2(t_0)+\dot{x}^2(t_0)+\ddot{x}^2(t_0))+\dddot{x}(t_0)\}$ for sufficiently large t, where K is a constant.

By the inequality (5.16) and Lemmas 4.1 and 4.2, we have established conditions (a), (b) and (d) of Theorem B. Condition (c) follows from (d) and hence, the completion of the proof of Theorem 3.1.

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