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ABSTRACT

In this paper, we investigated the Hadamard product of the integral operator of the form

$$F_\alpha(z) = \int_0^z \prod_{k=1}^{n} (f_k'(t))^\alpha$$

KEYWORDS: Univalent, Starlike, Convex.

2000 Mathematics Subject Classification: 30C45

INTRODUCTION

Let $H(u)$ denote the class of all analytic functions in the unit disk $U = \{z \in C: |z| < 1\}$. A is the set of analytic functions $f \in H(u)$ such that $f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \ z \in U$ and S is the set of functions $f \in A$ such that f is univalent in U. S^* is the class of functions in the unit disk defined by $S^* = \left\{ f \in H(u): f(0) = f'(0) - 1 = 0, R_e\left\{1 + \frac{zf'(z)}{f(z)}\right\} > 0, z \in U \right\}$.
S^c is the class of functions in U denoted by

$$S^c = \left\{ f \in H(u): f(0) = f'(0) - 1 = 0, R_e\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0, z \in U \right\}$$

S^* and S^c are the class of starlike and convex functions with respect to the origin respectively. For the complex valued functions, let

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$$f_1(z) = z + \sum_{n=2}^{\infty} a_{1n} z^n, \quad f_2(z) = z + \sum_{n=2}^{\infty} a_{2n} z^n$$

The Hadamard product is given by

$$f_1 * f_2 = f(z) = z + \sum_{n=2}^{\infty} a_{1n} a_{2n} z^n .$$

In 1973, Kudryashov investigated the maximum value of M such that the inequality

$$\frac{zf''(z)}{f'(z)} \leq M \qquad (1)$$

implies that f is univalent in U. He showed that if $M = 3{:}05 \ldots$ where M is the solution of the equations $8[M(M-2)^3]^{1/2} - 3(3-M)^2 = 12$ then f is univalent in U. Also, D. Breaz, S. Owa and N. Breaz considered the integral operator

$$F_{\alpha_1,\alpha_2,\ldots,\alpha_n}(z) = \int_0^z \left(f_1'(t)\right)^{\alpha_1} \ldots \left(f_n'(t)\right)^{\alpha_n} dt$$

showed that

$$\left|\frac{zf_i''(z)}{f_i'(z)}\right| \leq M \qquad (2)$$

is starlike where $M_1 = 2{:}8329 \ldots$ is the smallest root of equation $xsinx + cosx = 1/e.$

Furthermore, D. O. Makinde investigated the univalence, starlikeness and convexity of the integral operator

$$F_{x_1+iy_1, x_2+iy_2, \ldots, x_n+iy_n} = \int_0^z \prod_{k=1} \left(f_k'(t)\right)^{x_k+iy_k}$$

We now present our main results.

1. MAIN RESULTS

Theorem 1: Let $0 < \alpha < 1$ and let

$$\left|\frac{z(f_{1k} * f_{2k})''(z)}{(f_{1k} * f_{2k})'(z)}\right| \leq M_1,$$

then the integral operator

$$F_{1\alpha} * F_{2\alpha} = \int_0^z \prod_{k=1} \left((f_{1k} * f_{2k})'(t)\right)^{\alpha} dt$$

is univalent, $0 < n\alpha \leq 1, k \in \overline{1,n}, n \in N$.

Proof: **Let**

$$H(z) = \frac{(F_{1\alpha} * F_{2\alpha})''(z)}{(F_{1\alpha} * F_{2\alpha})'(z)}$$

But

$$(F_{1\alpha} * F_{2\alpha})'(z) = \prod_{k=1}^{n} \left((f_{1k} * f_{2k})'(z)\right)^{\alpha}$$

$$= \left((f_{11} * f_{21})'(z)\right)^{\alpha} \left((f_{12} * f_{22})'(z)\right)^{\alpha} \dots \left((f_{1n} * f_{2n})'(z)\right)^{\alpha}$$

Thus, for $k = 2$ we obtain

$$(F_{1\alpha} * F_{2\alpha})''(z) = \alpha\left((f_{11} * f_{21})'(z)\right)^{\alpha-1}\left((f_{11} * f_{21})''(z)\left((f_{11} * f_{21})'(z)\right)^{\alpha}\right) + \left((f_{11} * f_{21})'(z)\right)^{\alpha}\alpha\left((f_{12} * f_{22})'(z)\right)^{\alpha-1}\left((f_{12} * f_{22})'(z)\right)''$$

Simplify we obtain

$$z\frac{(F_{1\alpha} * F_{2\alpha})''(z)}{(F_{1\alpha} * F_{2\alpha})'(z)} = \alpha \sum_{k=1}^{2} z\frac{(f_{1k} * f_{2k}'')(z)}{(f_{1k} * f_{2k}')(z)}$$

Thus, inductively we obtain

$$z\frac{(F_{1\alpha} * F_{2\alpha})''(z)}{(F_{1\alpha} * F_{2\alpha})'(z)} = \alpha \sum_{k=1}^{n} z\frac{(f_{1k} * f_{2k}'')(z)}{(f_{1k} * f_{2k}')(z)}$$

$$\left|z\frac{(F_{1\alpha} * F_{2\alpha})''(z)}{(F_{1\alpha} * F_{2\alpha})'(z)}\right| \leq \alpha \sum_{k=1}^{n} \left|z\frac{(f_{1k} * f_{2k}'')(z)}{(f_{1k} * f_{2k}')(z)}\right|$$

$$\leq n\alpha M$$

$$\leq M$$

by the hypothesis that $0 < n\alpha \leq 1$

Thus, by (2) the integral operator

$$F_{1\alpha} * F_{2\alpha}(z) = \int_0^z \prod_{k=1}^{n} \left((f_{1k} * f_{2k})'(t)\right)^{\alpha} dt$$

is univalent U.

This concludes the proof of Theorem 1.

Corollary: Let $\alpha \in C$, $|\alpha| < 1$ and let

$$\left|\frac{z(f_{1k} * f_{2k})''(z)}{(f_{1k} * f_{2k})'(z)}\right| \leq M$$

then the integral operator

$$F_{1\alpha} * F_{2\alpha} = \int_0^z \prod_{k=1}^{n} \left((f_{1k} * f_{2k})'(t)\right)^{\alpha} dt$$

is univalent.

Theorem 2. Let $\alpha \in R$, $\alpha > 0$. Suppose $(f_{1k} * f_{2k})(z)$ is convex for all $k \in \{1,2,\dots,n\}$. Then the integral operator

$$F_{1\alpha} * F_{2\alpha}(z) = \int_0^z \prod_{k=1}^{n} \left((f_{1k} * f_{2k})'(t)\right)^{\alpha} dt$$

is convex in U.

Proof:

$$\left\{1 + \frac{(F_{1\alpha} * F_{2\alpha})''(z)}{(F_{1\alpha} * F_{2\alpha})'(z)}\right\} = \left\{\sum_{k=1}^{n} \alpha z \frac{(f_{1k} * f_{2k})''(z)}{((f_{1k} * f_{2k})')(z)}\right\} + 1$$

$$\geq z \frac{(f_{1k} * f_{2k})''(z)}{((f_{1k} * f_{2k})')(z)} + 1$$

by hypothesis $(f_{1k} * f_{2k})(z)$ is convex that is

$$Re\left\{z \frac{(f_{1k} * f_{2k})''(z)}{(f_{1k} * f_{2k})'(z)} + 1\right\} > 0$$

Thus,

$$Re\left\{1+\frac{(f_{1k}*f_{2k})''(z)}{(f_{1k}*f_{2k})'(z)}\right\}=Re\left\{\sum_{k=1}^{n}\alpha z\frac{(f_{1k}*f_{2k})''(z)}{(f_{1k}*f_{2k})'(z)}+1\right\}$$
$$\geq Re\left\{z\frac{(f_{1k}*f_{2k})''(z)}{(f_{1k}*f_{2k})'(z)}+1\right\}>0$$

Thus the integral operator

$$F_{1\alpha}*F_{2\alpha}=\int_0^z\prod_{k=1}^{n}\left((f_{1k}*f_{2k})'(t)\right)^{\alpha}dt$$

is convex in U.

This concludes the proof of Theorem 2.

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