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ON THE BOUNDEDNESS AND THE STABILITY OF SOLUTION TO THIRD ORDER NON-LINEAR DIFFERENTIAL EQUATIONS *

B.S. Ogundare, G.E. Okecha
(*Dept of Pure and Applied Math., University of Fort Hare, Alice, 5700 RSA*)

Abstract
In this paper we investigate the global asymptotic stability, boundedness as well as the ultimate boundedness of solutions to a general third order nonlinear differential equation, using complete Lyapunov function.

Keywords complete Lyapunov function; global asymptotic stability; third order non-linear differential equations

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1 Introduction

The concept of stability as well as the boundedness of solution is very important in the theory and applications of differential equations. It is also established so far that the most effective method to study these concepts (especially stability) for non-linear differential equations is the Lyapunov second method.

Consider the equation

$$\dddot{x} + \varphi(x, \dot{x})\ddot{x} + f(x, \dot{x}) = p(t; x, \dot{x}, \ddot{x}), \tag{1.1}$$

where φ, f and p are continuous, and depend on the arguments displayed explicitly. In addition, they are such that the existence, uniqueness and continuous dependence on initial condition is guaranteed.

Boundedness and stability properties of solutions to various forms of equation (1.1) had received considerable attention. Many of these are summarised in [8]. In [6], 2 variants or classes of equation (1.1) were considered. Also in [7], the author re-visited the problem of Barbashin [2], where the equation above was considered. Barbashin [2], came up with interesting result on the equation by considering the equation as a general third order nonlinear differential equation. His results could not handle some of the special cases (or variants) of the equation as we have in [4-7]. In an attempt handle and accommodate almost all the classes (and variants) of equation (1.1), Qian [7], came up with results which simplified the theory of Barbashin and thereby making the result applicable to wider classes or forms of equation (1.1).

Our aim in this paper is to further simplify the theorem of Barbashin [2] and Qian [7] by extending results in [6]. Here we discuss the boundedness and ultimately boundedness of the solution to equation (1.1) on a real line.

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As in [6], equation (1.1) is better handled as a system of three-coupled first order equations by letting;

$$\begin{aligned}\dot{x} &= y,\\ \dot{y} &= z,\\ \dot{z} &= -\varphi(x,y)z - f(x,y) + p(t;x,\dot{x},\ddot{x})\end{aligned} \tag{1.2}$$

In this study, we use a single complete Lyapunov function to achieve our result.

For expository reasons, we give the following definitions:

Definition 1.1 Let

$$\dot{x} = f(t,x) \tag{1.3}$$

be a system of n-first order differential equations, a Lyapunov function V defined as $V : I \times \Re^n \rightarrow \Re$ is said to be COMPLETE if for $X \in \Re^n$

(i) $V(t,X) \geq 0$;

(ii) $V(t,X) = 0$, if and only if $X = 0$ and

(iii) $\dot{V}(t,X)|_{1.2} \leq -c|X|$ where c is any positive constant and $|X|$ given by $|X| = (\sum\limits_{i=1}^{n} x_i^2)^{\frac{1}{2}}$ such that $|X| \rightarrow \infty$ as $X \rightarrow \infty$.

Definition 1.2 A Lyapunov function V defined as $V : I \times \Re^n \rightarrow \Re$ is said to be INCOMPLETE if for $X \in \Re^n$ (i) and (ii) of the above definition is satisfied. In addition,

(iii) $\dot{V}(t,X)|_{1.2} \leq -c|X|_*$ where c is any positive constant and $|X|_*$ given by $|X|_* = (\sum\limits_{i=1}^{j} x_i^2)^{\frac{1}{2}}$, where by j $(i \leq j < n)$ we mean that not all the variables (otherwise called the trajectories) are necessarily involved such that $|X|_* \rightarrow \infty$ as $X \rightarrow \infty$.

The particular case, according to this work, is when $n = 3$. Therefore we refer to (1.2) as our system of 3-first order differential equations and the definitions can now be put as follows:

Definition 1.3 A Lyapunov function V defined as $V : I \times \Re^3 \rightarrow \Re$ is said to be COMPLETE if for $X \in \Re^3$

(i) $V(t,X) \geq 0$;

(ii) $V(t,X) = 0$, if and only if $X = 0$ and

(iii) $\dot{V}(t,X)|_{1.2} \leq -c|X|$ where c is any positive constant and $|X|$ given by $|X| = (\sum\limits_{i=1}^{3} x_i^2)^{\frac{1}{2}}$ such that $|X| \rightarrow \infty$ as $X \rightarrow \infty$.

Definition 1.4 A Lyapunov function V defined as $V : I \times \Re^3 \rightarrow \Re$ is said to be INCOMPLETE if for $X \in \Re^3$ (i) and (ii) of the above definition is satisfied. In addition,

(iii) $\dot{V}(t,X)|_{1.2} \leq -c|X|_*$ where c is any positive constant and $|X|_*$ given by $|X|_* = (x^2+y^2)^{\frac{1}{2}}$, or $|X|_* = (x^2+z^2)^{\frac{1}{2}}$, or $|X|_* = (y^2+z^2)^{\frac{1}{2}}$, or $|X|_* = (x^2)^{\frac{1}{2}}$, or $|X|_* = (y^2)^{\frac{1}{2}}$, or $|X|_* = (z^2)^{\frac{1}{2}}$ such that $|X|_* \rightarrow \infty$ as $X \rightarrow \infty$.

2 Formulation of Results

We will consider equation (1.1) in two major ways and have the following theorems to prove.

Case 1 When $p(t, x, \dot{x}, \ddot{x}) \equiv 0$. This may be considered as the homogeneous case.

Theorem 2.1 *Let f and φ be continuous and let $I_0 = [\delta, J]$ where $J = \beta\kappa\varepsilon(1-\varepsilon)$, δ, β, κ and ε are positive constants. Moreover, let the following conditions hold:*

(i) $f_x = \frac{f(x,y)-f(0,y)}{x} \in I_0 = \alpha, \ x \neq 0$,

(ii) $f_y = \frac{f(x,y)-f(x,0)}{y} \in I_0 = \beta, \ y \neq 0$,

(iii) $f(0, y) = f(x, 0) = 0$ *and*

(iv) $|\varphi(x, y)| \leq \kappa$.

Then the trivial solution to equation (1.1) *is globally asymptotically stable.*

Case 2 When $p(t, x, \dot{x}, \ddot{x}) \neq 0$. The non-homogeneous case

Theorem 2.2 *Suppose that the conditions of the Theorem 2.1 are satisfied, further, $|p(t; x, \dot{x}, \ddot{x})| \leq A$, then there exists a constant $\mu(0 < \mu < \infty)$ depending only on β, δ and κ such that every solution of equation* (1.1) *satisfies*

$$x^2(t) + \dot{x}^2(t) + \ddot{x}^2(t) \leq e^{-\frac{1}{2}\mu t}\Big\{A_1 + A_2 \int_{t_0}^{t} |P(\tau)| e^{\frac{1}{2}\mu\tau} \mathrm{d}\tau\Big\}^2$$

for all $t \geq t_0$, where the constant $A_1 > 0$, depends on β, δ, κ as well as on $t_0, x(t_0), \dot{x}(t_0)$ and $\ddot{x}(t_0)$; and the constant $A_2 > 0$ depends on β, δ and κ only.

Theorem 2.3 *Following the assumptions of Theorem 2.2 and taking $|p(t; x, \dot{x}, \ddot{x})| = (|x| + |y| + |z|)\phi(t)$, where $\phi(t)$ is a non-negative and continuous function of t and satisfies $\int_0^t \phi(s)\mathrm{d}s \leq M < \infty$ where M is a positive constant. Then, there exists a constant K_0, which depends on M, K_1, K_2 and t_0, such that every solution $x(t)$ to equation* (1.1) *satisfies*

$$|x(t)| \leq K_0, \quad |\dot{x}(t)| \leq K_0, \quad |\ddot{x}(t)| \leq K_0$$

for sufficiently large t.

Notations Throughout this paper, $K, K_0, \cdots, K_{11}$ denote finite positive constants whose magnitudes depend only on the functions ϕ, f and P, as well as constants a, κ, β, Δ and δ. But they are independent of the solutions to (1.1). Meanwhile, the K_i occurred each time is not necessarily the same, but each $K_i, i = 1, 2, \cdots$ retains its identity throughout.

3 The Function $V(x, y, z)$

The main tool used in the proof of the theorems is the function $V = V(x, y, z)$, which we obtained below after lengthy algebraic computations

$$\begin{aligned} 2V = \frac{a\delta}{\Delta}\{&[\beta^2(1-\varepsilon)^2]x^2 + [(1-\varepsilon)(\kappa^2 - \beta(1-\varepsilon)) + \beta]y^2 + z^2 \\ &+ 2\kappa\beta(1-\varepsilon)^2 xy + 2(1-\varepsilon)^2\beta xz + 2\kappa(1-\varepsilon)yz\}, \end{aligned} \tag{3.1}$$

where $a, \beta, \varepsilon, \Delta, \kappa$ and δ are all positive for all x, y, z. $\delta > 1$ and $\Delta = \alpha\beta(\delta-1)(1-\varepsilon)^2$.

The following lemmas are to prove that $V(x, y, z)$ is indeed a Lyapunov function.

Lemma 3.1 *Subject to the assumptions of Theorem 2.1 there exist positive constants $K_i = K_i(a, \beta, \varepsilon, \Delta, \kappa, \delta)$, $i = 1, 2$ such that*

$$K_1(x^2 + y^2 + z^2) \leq V(x, y, z) \leq K_2(x^2 + y^2 + z^2). \tag{3.2}$$

Proof Clearly, $V(0, 0, 0) = 0$.

By rearranging (3.1) we have

$$2V = \frac{a\delta}{\Delta}\{[\beta(1-\varepsilon)x+\kappa(1-\varepsilon)y+z]^2+\beta^2(1-\varepsilon)^2x^2+\varepsilon[(1-\varepsilon)\kappa+\beta\varepsilon]y^2-\varepsilon\beta(1-\varepsilon)xz\}, \quad (3.3)$$

$$2V = \frac{a\delta}{\Delta}\Big\{[\beta(1-\varepsilon)x+\kappa(1-\varepsilon)y+z]^2+\beta^2\varepsilon(1-\varepsilon)^2x^2-\beta\varepsilon(1-\varepsilon)\Big(x+\frac{1}{2}z\Big)^2$$
$$+\varepsilon[\kappa(1-\varepsilon)+\beta\varepsilon]y^2+\beta\frac{\varepsilon^2(1-\varepsilon)}{4}z^2\Big\}, \quad (3.4)$$

and from equation (3.4), we obtain

$$2V \ge \frac{a\delta}{\Delta}\Big\{\beta^2\varepsilon(1-\varepsilon)^2x^2+\varepsilon[\kappa(1-\varepsilon)+\beta\varepsilon]y^2+\beta\frac{\varepsilon^2(1-\varepsilon)}{4}z^2\Big\} \quad (3.5)$$

and this reduces to

$$V \ge K_1(x^2+y^2+z^2), \quad (3.6)$$

where

$$K_1 = \frac{a\delta}{2\Delta}\cdot\min\Big\{\beta^2\varepsilon(1-\varepsilon)^2, \varepsilon[\kappa(1-\varepsilon)+\beta\varepsilon], \beta\frac{\varepsilon^2(1-\varepsilon)}{4}\Big\}$$

By the Schwartz inequality,

$$|xy| \le \frac{1}{2}|x^2+y^2|,$$

and so from equation (3.1) we have

$$2V \le \frac{a\delta}{\Delta}\{[\beta^2(1-\varepsilon)^2]x^2+\left[(1-\varepsilon)(\kappa^2-\beta(1-\varepsilon))+\beta\right]y^2+z^2$$
$$+\kappa\beta(1-\varepsilon)^2(x^2+y^2)+(1-\varepsilon)^2\beta(x^2+z^2)+\kappa(1-\varepsilon)(y^2+z^2)\}, \quad (3.7)$$

$$2V \le \frac{a\delta}{\Delta}\{\beta^2\varepsilon^2(1+\beta+\kappa)x^2+(1-\varepsilon)\left[\kappa(\kappa+1)+\beta(1-\varepsilon)(\kappa-1)\right]y^2$$
$$+\left[1+(1-\varepsilon)(\kappa+\beta(1-\varepsilon))\right]z^2\}, \quad (3.8)$$

and which reduces that

$$V \le K_2(x^2+y^2+z^2), \quad (3.9)$$

where

$$K_2 = \frac{\delta}{2\Delta}\cdot\max\{\beta^2\varepsilon^2(1+\beta+\kappa), (1-\varepsilon)\left[\kappa(\kappa+1)+\beta(1-\varepsilon)(\kappa-1)\right],$$
$$\left[1+(1-\varepsilon)(\kappa+\beta(1-\varepsilon))\right]\}.$$

Combining equations (3.6) with (3.9), we have

$$K_1(x^2+y^2+z^2) \le V(x,y,z) \le K_2(x^2+y^2+z^2). \quad (3.10)$$

This proves Lemma 3.1.

Lemma 3.2 *Suppose that the conditions of Theorem* 2.1 *hold, then there are positive constants* $K_3 = K_3(a,\Delta,\delta)$, *such that for any solution* (x,y,z) *to system* (1.2),

$$\dot V|_{(1.2)} \equiv \frac{\mathrm{d}}{\mathrm{d}t}V\Big|_{(1.2)}(x,y,z) \le -K_3(x^2+y^2+z^2). \quad (3.11)$$

Proof From equations (1.1) and system (1.2) we have

$$V|_{(1.2)} = \frac{\partial V}{\partial x}\dot{x} + \frac{\partial V}{\partial y}\dot{y} + \frac{\partial V}{\partial z}\dot{z} = \frac{\partial V}{\partial x}y + \frac{\partial V}{\partial y}z + \frac{\partial V}{\partial z}(-\varphi(x,y)z - f(x,y)),$$

which then gives

$$\begin{aligned}\dot{V} = \frac{a\delta}{\Delta}\{&[\beta^2(1-\varepsilon)^2]xy + \left[(1-\varepsilon)(\kappa^2 - \beta(1-\varepsilon)) + \beta\right] yz + z(-\varphi(x,y)z - f(x,y)) \\ &+\kappa\beta(1-\varepsilon)^2(y^2 + xz) + (1-\varepsilon)^2\beta[yz + x(\varphi(x,y)z - f(x,y))] \\ &+\kappa(1-\varepsilon)[z^2 + y(\varphi(x,y)z - f(x,y))]\}.\end{aligned} \tag{3.12}$$

Then by the conditions on $f(x,y)$, that is, $\frac{f(x,y)-f(0,y)}{x} = f_x$, and $\frac{f(x,y)-f(x,0)}{y} = f_y$, and after the simplification process, we have

$$\dot{V} = -\frac{a\delta}{\Delta}\{x^2 + y^2 + z^2\}. \tag{3.13}$$

Let $K_3 \leq \frac{a\delta}{\Delta}$, then $\dot{V} \leq -K_3(x^2 + y^2 + z^2)$. This completes the proof of Lemma 3.2.

Lemma 3.3 *Suppose that the conditions of Theorem 2.2 hold, then there exist positive constants* $K_j = K_j(a, \beta, \varepsilon, \kappa, \Delta, \delta)(j = 4, 5)$, *such that for any solution* (x, y, z) *to system* (1.2),

$$\dot{V}|_{(1.2)} \equiv \frac{\mathrm{d}}{\mathrm{d}t}V\Big|_{(1.2)}(x,y,z) \leq -K_4(x^2 + y^2 + z^2) + K_5(|x| + |y| + |z|)\,|p(t; x, \dot{x}, \ddot{x})|. \tag{3.14}$$

Proof Following the same reasoning as that in Lemma 3.2, given $p \neq 0$, $p(t; x, y, z) = P(t)$, we have that

$$\begin{aligned}\dot{V} = \frac{a\delta}{\Delta}\{&[\beta^2(1-\varepsilon)^2]xy + \left[(1-\varepsilon)(\kappa^2 - \beta(1-\varepsilon)) + \beta\right] yz + z(-\varphi(x,y)z - f(x,y) + P(t)) \\ &+\kappa\beta(1-\varepsilon)^2(y^2 + xz) + (1-\varepsilon)^2\beta[yz + x(\varphi(x,y)z - f(x,y) + P(t))] \\ &+\kappa(1-\varepsilon)[z^2 + y(\varphi(x,y)z - f(x,y) + P(t))]\}.\end{aligned} \tag{3.15}$$

Also, by the conditions on $f(x,y)$ and $\varphi(x,y)$

$$\begin{aligned}\dot{V} &= -\frac{a\delta}{\Delta}\{x^2 + y^2 + z^2 - ((1-\varepsilon^2\beta)x + \kappa(1-\varepsilon)y + z)P(t)\} \\ &\leq -\frac{a\delta}{\Delta}\{x^2 + y^2 + z^2 - K_4(|x| + |y| + |z|)P(t)\},\end{aligned} \tag{3.16}$$

where $K_4 = \max\{(1-\varepsilon)^2\beta, \kappa(1-\varepsilon), 1\}$. Hence

$$\dot{V} \leq -K_3(x^2 + y^2 + z^2) + K_5(|x| + |y| + |z|)\,|P(t)|, \tag{3.17}$$

where $K_5 = \frac{K_4 a\delta}{\Delta}$.

Since $(|x| + |y| + |z|) \leq \sqrt{3}(x^2 + y^2 + z^2)^{\frac{1}{2}}$, inequality (3.17) becomes

$$\frac{\mathrm{d}V}{\mathrm{d}t} \leq -K_4(x^2 + y^2 + z^2) + K_6(x^2 + y^2 + z^2)^{\frac{1}{2}}\,|P(t)|, \tag{3.18}$$

where $K_6 = \sqrt{3}K_5$ and $K_4 = K_3$. This completes the proof of Lemma 3.3.

From the proofs of the lemmas, we establish that function $V(x, y, z)$ is a Lyapunov function.

4 Proof of the Main Results

We now give the proof of the Theorems stated in Section 2 of this paper.

Proof of Theorem 2.1 It follows from the proof of the Lemmas 3.1 and 3.2 that the trivial solution to equation (1.1) is globally asymptotically stable. That is, every solution $(x(t), \dot{x}(t), \ddot{x}(t))$ to system (1.2) satisfies $x^2(t) + \dot{x}^2(t) + \ddot{x}^2(t) \to 0$ as $t \to \infty$.

Proof of Theorem 2.2 From inequality (3.18),

$$\frac{dV}{dt} \leq -K_3(x^2 + y^2 + z^2) + K_6(x^2 + y^2 + z^2)^{\frac{1}{2}} |P(t)| .$$

At the same time, from inequality (3.6), we have

$$(x^2 + y^2 + z^2)^{\frac{1}{2}} \leq \left(\frac{2V}{K_1}\right)^{\frac{1}{2}}.$$

Let $K_4(x^2 + y^2 + z^2) = K_4 \cdot \frac{V}{K_1}$, we have

$$\frac{dV}{dt} \leq -K_7 V + K_8 V^{\frac{1}{2}} |P(t)| , \tag{4.1}$$

where $K_7 = \frac{K_4}{K_2}$ and $K_8 = \frac{K_6}{K_2^{\frac{1}{2}}}$. So

$$\dot{V} \leq -2K_9 V + K_8 V^{\frac{1}{2}} |P(t)| , \tag{4.2}$$

where $K_9 = \frac{1}{2}K_7$. Therefore

$$\dot{V} + K_9 V \leq -K_9 V + K_8 V^{\frac{1}{2}} |P(t)| \leq K_8 V^{\frac{1}{2}} \{ |P(t)| - K_{10} V^{\frac{1}{2}} \}, \tag{4.3}$$

where $K_{10} = \frac{K_9}{K_8}$. Thus inequality (4.3) becomes

$$\dot{V} + K_9 V \leq K_8 V^{\frac{1}{2}} V^* \tag{4.4}$$

where

$$V^* = |P(t)| - K_{10} V^{\frac{1}{2}} \leq V^{\frac{1}{2}} |P(t)| \leq |P(t)| \tag{4.5}$$

When $|P(t)| \leq K_{10} V^{\frac{1}{2}}$,

$$V^* \leq 0, \tag{4.6}$$

and when $|P(t)| \geq K_{10} V^{\frac{1}{2}}$,

$$V^* \leq |P(t)| \cdot \frac{1}{K_{10}}. \tag{4.7}$$

On substituting inequality (4.6) into (4.3), we have

$$\dot{V} + K_9 V \leq K_{11} V^{\frac{1}{2}} |P(t)|$$

where $K_{11} = \frac{K_8}{K_{10}}$. This implies that

$$V^{-\frac{1}{2}} \dot{V} + K_9 V^{\frac{1}{2}} \leq K_{11} |P(t)| . \tag{4.8}$$

Multiplying both sides of inequality (4.8) by $e^{\frac{1}{2}K_9 t}$, we have

$$e^{\frac{1}{2}K_9 t} \{ V^{-\frac{1}{2}} \dot{V} + K_9 V^{\frac{1}{2}} \} \leq e^{\frac{1}{2}K_9 t} K_{11} |P(t)| . \tag{4.9}$$

that is,

$$2\frac{\mathrm{d}}{\mathrm{d}t}\{V^{\frac{1}{2}}e^{\frac{1}{2}K_9t}\} \le e^{\frac{1}{2}K_9t}K_{11}\,|P(t)| \tag{4.10}$$

Integrating both sides of (4.10) from t_0 to t, gives

$$\{V^{\frac{1}{2}}e^{\frac{1}{2}K_9\gamma}\}_{t_0}^{t} \le \int_{t_0}^{t}\frac{1}{2}e^{\frac{1}{2}K_9\tau}K_{11}|P(\tau)|\mathrm{d}\tau, \tag{4.11}$$

which implies that

$$\{V^{\frac{1}{2}}(t)\}e^{\frac{1}{2}K_9t} \le V^{\frac{1}{2}}(t_0)e^{\frac{1}{2}K_9t_0} + \frac{1}{2}K_{11}\int_{t_0}^{t}|P(\tau)|\,e^{\frac{1}{2}K_9\tau}\mathrm{d}\tau,$$

or

$$V^{\frac{1}{2}}(t) \le e^{-\frac{1}{2}K_9t}\Big\{V^{\frac{1}{2}}(t_0)e^{\frac{1}{2}K_9t_0} + \frac{1}{2}K_{11}\int_{t_0}^{t}|P(\tau)|\,e^{\frac{1}{2}K_9\tau}\mathrm{d}\tau\Big\}$$

Using (3.9) and (3.10) we have

$$\begin{aligned}K_1(x^2(t)+\dot{x}^2(t)+\ddot{x}^2(t)) \le e^{-\frac{1}{2}K_9t}\Big\{&K_2(x^2(t_0)+\dot{x}^2(t_0)+\ddot{x}^2(t_0))e^{\frac{1}{2}K_9t_0}\\ &+\frac{1}{2}K_{11}\int_{t_0}^{t}|P(\tau)|\,e^{\frac{1}{2}K_9\tau}\mathrm{d}\tau\Big\}^2 \quad \text{for all } t \ge t_0.\end{aligned} \tag{4.12}$$

Thus,

$$\begin{aligned}x^2(t)+\dot{x}^2(t)+\ddot{x}^2(t) &\le \frac{1}{K_1}\Big\{e^{-\frac{1}{2}K_9t}\Big[K_2(x^2(t_0)+\dot{x}^2(t_0)+\ddot{x}^2(t_0))e^{\frac{1}{2}K_9t_0}\\ &\quad +\frac{1}{2}K_{11}\int_{t_0}^{t}|P(\tau)|\,e^{\frac{1}{2}K_9\tau}\mathrm{d}\tau\Big]^2\Big\}\\ &\le e^{-\frac{1}{2}K_9t}\Big[A_1 + A_2\int_{t_0}^{t}|P(\tau)|\,e^{\frac{1}{2}K_9\tau}\mathrm{d}\tau\Big]^2,\end{aligned} \tag{4.13}$$

where A_1 and A_2 are constants depending on $\{K_1, K_2, (x^2(t_0)+\dot{x}^2(t_0)+\ddot{x}^2(t_0))\}$ and $\{K_1, K_{11}\}$ respectively.

By substituting $K_9 = \mu$ into inequality (4.13), we have

$$x^2(t)+\dot{x}^2(t)+\ddot{x}^2(t) \le e^{-\frac{1}{2}\mu t}\Big[A_1 + A_2\int_{t_0}^{t}|P(\tau)|\,e^{\frac{1}{2}\mu\tau}\mathrm{d}\tau\Big]^2,$$

which completes the proof.

Proof of Theorem 2.3 From the function V defined above and the conditions of Theorem 2.3, the conclusion of Lemma 3.1 can then be obtained, as

$$V \ge K_1(x^2+y^2+z^2), \tag{4.14}$$

and since $P \ne 0$ we can revise the result of Lemma 3.2, that is,

$$\dot{V} \le -K_4(x^2+y^2+z^2) + K_5(|x|+|y|+|z|)\,|P(t)|,$$

and we obtain

$$\dot{V} \le K_5(|x|+|y|+|z|)^2 r(t) \tag{4.15}$$

By Schwartz inequalities on (4.15), we have

$$\dot{V} \leq K_{11}(x^2 + y^2 + z^2)r(t), \tag{4.16}$$

where $K_{11} = 3K_5$. From inequalities (4.14) and (4.16) we have

$$\dot{V} \leq K_{11} V r(t). \tag{4.17}$$

Integrating equation (4.17) from 0 to t, we obtain

$$V(t) - V(0) \leq K_{12} \int_0^t V(s)r(s)\mathrm{d}s, \tag{4.18}$$

where $K_{12} = \frac{K_{11}}{K_1} = \frac{3K_5}{K_1}$. Using the condition on $p(t;x,y,z)$ as stated in Theorem 2.3 we have

$$V(t) \leq V(0) + K_{11} \int_0^t V(s)r(s)\mathrm{d}s. \tag{4.19}$$

By Grownwall-Bellman inequality, inequality (4.19) yields,

$$V(t) \leq V(0) \exp\left(K_{12} \int_0^t r(s)\mathrm{d}s\right). \tag{4.20}$$

This completes the proof of Theorem 2.3.

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