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On a four-parameter type I generalized half logistic distribution

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Abstract

In this paper, we derived a four-parameter type I generalized half logistic distribution. The cumulative distribution function, the survival and hazard function, the moments, the $100p$-percentage point and the mode of the generalized distribution are established. Two theorems that characterized the probability distribution function are stated and proved.

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1 Introduction

A random variable X is said to have an half logistic distribution if its probability density function is

$$f_X(x) = \frac{2exp(x)}{(1+exp(x))^2}, \quad x > 0. \tag{1.1}$$

Balakrishnan (1985) established some recurrence relations for the moments and product moments of order statistics for half logistic distribution. If a location parameter μ and a

*Currently doing post-doctoral study at: Department of Statistics and Actuarial Science, Stellenbosch University, Stellenbosch, South Africa. Mobile phone: +27 78 161 7025 scale parameter σ are introduced in the equation (1.1). the density of the random variable X becomes

$$f_X(x;\mu,\sigma) = \frac{2exp(\frac{x-\mu}{\sigma})}{\sigma(1+exp(\frac{x-\mu}{\sigma}))^2}, \quad x>0,\ \mu\geq 0,\ \sigma>0. \tag{1.2}$$

Balakrishnan and Puthenpura (1986) obtained the best linear unbiased estimates of the location and scale parameters μ and σ respectively of the half logistic distribution through linear functions of order statistics and they also tabulated the values of the variance and covariance of these estimators. Balakrishnan and Wong (1991) obtained approximate maximum likelihood estimates of the location and scale parameters of the half logistic distribution with Type-II Right-Censoring. Olapade (2003) stated and proved some theorems that characterized the half logistic distribution.

2 Four-parameter type I generalized half logistic distribution

Olapade (2004) obtained a generalized form of the logistic distribution which contains four parameters as

$$f_X(x;\mu,\sigma,\lambda,p) = \frac{p\lambda^p}{\sigma}\frac{e^{-(\frac{x_i-\mu}{\sigma})}}{[\lambda+e^{-(\frac{x_i-\mu}{\sigma})}]^{(p+1)}}, \quad -\infty<x<\infty, \tag{2.0}$$

where μ is the location parameter, σ is the scale parameter, p is the shape parameter and λ is a shift parameter. In this research, we want to obtain a generalized half logistic form of the generalized logistic distribution in equation (2.0) which we shall call a four-parameter type I generalized half logistic distribution and study its theories and properties. A similar work has been done for a four-parameter type II generalized half logistic distribution in Olapade (2009). Let us consider the theorem below.

Theorem 2.1: *Let X be a continuously distributed random variable with density function $f_X(x)$. The random variable $Y = \ln(2e^X - \lambda)$ has a generalized half logistic distribution if X has an exponential distribution with parameter q where $\lambda > 0$ is a constant.*

Proof: If X is exponentially distributed with parameter q,

$$f_X(x) = qe^{-qx}, \quad 0<x<\infty,\ q>0. \tag{2.1}$$

Let $y=\ln(2e^x-\lambda)$, by transformation of random variable $x=\ln(\frac{\lambda-e^y}{2})$ and $dx/dy = e^y/(\lambda+e^y)$.

$$f_Y(y;q,\lambda) = |\frac{dx}{dy}|f_X(y) = \frac{Ke^y}{(\lambda+e^y)^{q+1}} \tag{2.2}$$

where $K = q(\lambda+1)^q$. Then

$$f_Y(y;q,\lambda) = \frac{q(\lambda+1)^q e^y}{(\lambda+e^y)^{q+1}}, \quad 0 \le y < \infty, \quad q > 0, \ \lambda > 0. \tag{2.3}$$

If we introduce the location parameter μ and the scale parameter σ in equation (2.3), we have

$$f_Y(y;\mu,\sigma,q,\lambda) = \frac{q(\lambda+1)^q e^{(\frac{y-\mu}{\sigma})}}{\sigma(\lambda+e^{(\frac{y-\mu}{\sigma})})^{q+1}}, \ y > 0, \ \mu > 0, \ \sigma > 0, \ q > 0, \ \lambda > 0. \tag{2.4}$$

This probability density function in equation (2.4) is what we call a four-parameter type I generalized half logistic distribution. It gives the half logistic version of the extended type I generalized logistic distribution in equation (2.0). In the rest of this paper, we shall assume that $\mu = 0$ and $\sigma = 1$ without loss of generality.

3 The cumulative distribution function, the survival function and the hazard function of the four-parameter type I generalized half logistic distribution

The cumulative distribution function of the four-parameter type I generalized half logistic distribution is obtained from equation (2.3) as

$$F_Y(y;q,\lambda) = \int_0^y f_Y(t;q,\lambda)dt = q(\lambda+1)^q \int_0^y \frac{e^y}{(\lambda+e^y)^{q+1}}dt$$

$$= 1 - (\frac{\lambda+1}{\lambda+e^y})^q. \tag{3.1}$$

The probability that the four-parameter type I generalized half logistic random variable X lies in an interval (α_1, α_2) is given as

$$pr(\alpha_1 < Y < \alpha_2) = F_Y(\alpha_2) - F_Y(\alpha_1)$$

$$= (\lambda+1)^q[(\lambda+e^{\alpha_1})^{-q} - (\lambda+e^{\alpha_2})^{-q}], \ \ \forall\, \alpha_1 < \alpha_2. \tag{3.2}$$

Hence for any given value of the parameter q and an interval (α_1, α_2), the probability $pr(\alpha_1 < Y < \alpha_2)$ can be easily computed for a random variable that has the four-parameter type I generalized half logistic distribution.

If Y is the lifetime of an object, then the survival function $S(y)$ of Y is $S(y) = pr(Y > y) = (\lambda+1)^q(\lambda+e^y)^{-q}$. This is the probability that an object whose life time is a random variable Y that follows the four-parameter type I generalized half logistics distribution will survive beyond time y.

The hazard function of the random variable Y is $H(y) = f(y)/(1 - F(y))$, then if Y follows the four-parameter type I generalized half logistics distribution, its hazard function is $H(y) = qe^y/(\lambda + e^y)$. When $\lambda > q$, $H(y)$ asymptotically tends to unity as $y \to \infty$. When $\lambda \leq q$, $H(y)$ asymptotically tends to q as $y \to \infty$.

4 Moments of the Four-Parameter Type I Generalized Half Logistic Distribution

Considering the Four-Parameter Type I generalized half logistic distribution function $f_Y(y; q, \lambda)$ given in equation (2.3). The n^{th} moment of Y is

$$E[Y^n] = \int y^n f_Y(y; q, \lambda)dy = q(\lambda+1)^q \int_0^\infty \frac{y^n e^y}{(\lambda + e^y)^{q+1}}dy \tag{4.1}$$

The tables below shows a tabulated value of $E[Y^n]$ for $q = 0.25, 0.50, 0.75, 1.00, 1.50, 2.00, 2.50, 3.00, 3.50, 4.00, 4.50$ and 5.00 when $\lambda = 1.5, 2.0, 2.5$ and 3.0. These values can be used to compute the mean, variance, skewness and kurtosis for the four-parameter type I generalized half logistic distribution using the following relations:

$$\begin{aligned} \mu_1 &= \nu_1 \\ \mu_2 &= \nu_2 - \nu_1^2 \\ \mu_3 &= \nu_3 - 3\nu_2\nu_1 + 2\nu_1^3 \\ \mu_4 &= \nu_4 - 4\nu_3\nu_1 + 6\nu_2\nu_1^2 - 3\nu_1^4 \end{aligned} \tag{4.2}$$

where ν_i is the i^{th} moment $E[Y^i]$ and μ_1 = the mean, μ_2 = the variance, skewness $\beta_1 = \mu_3^2/\mu_2^3$ and the measure of kurtosis $\beta_2 = \mu_4/\mu_2^2$.

Tables of moments of the four-parameter type I generalized half logistic distribution.

Table 4.1 : $\lambda = 1.5$

q	$E[Y]$	$E[Y^2]$	$E[Y^3]$	$E[Y^4]$
0.2500	4.5642	34.7364	353.6109	4318.2612
0.5000	2.6623	11.8306	73.3654	584.4065
0.7500	1.9202	6.0704	26.3340	145.8600
1.0000	1.5271	3.8246	12.9783	55.5376
1.5000	1.1065	2.0170	4.9265	14.9469
2.0000	0.8786	1.2838	2.5076	6.0422
2.5000	0.7330	0.9028	1.4877	3.0159
3.0000	0.6310	0.6754	0.9698	1.7117
3.5000	0.5550	0.5273	0.6739	1.0593
4.0000	0.4961	0.4246	0.4906	0.6978
4.5000	0.4489	0.3502	0.3700	0.4819
5.0000	0.4101	0.2943	0.2869	0.3453

Table 4.2 : $\lambda = 2.0$

q	$E[Y]$	$E[Y^2]$	$E[Y^3]$	$E[Y^4]$
0.2500	4.7148	36.2222	369.7161	4518.2677
0.5000	2.8059	12.7524	79.8509	638.5948
0.7500	2.0513	6.7087	29.6361	165.7171
1.0000	1.6479	4.3102	15.0144	65.2710
1.5000	1.2115	2.3404	5.9483	18.5534
2.0000	0.9719	1.5216	3.1257	7.8215
2.5000	0.8172	1.0877	1.9014	4.0367
3.0000	0.7078	0.8246	1.2650	2.3550
3.5000	0.6258	0.6508	0.8943	1.4917
4.0000	0.5617	0.5291	0.6606	1.0025
4.5000	0.5101	0.4399	0.5046	0.7045
5.0000	0.4676	0.3723	0.3958	0.5127

Table 4.3 : $\lambda = 2.5$

q	$E[Y]$	$E[Y^2]$	$E[Y^3]$	$E[Y^4]$
0.2500	4.8434	37.5175	383.8634	4694.4094
0.5000	2.9300	13.5748	85.7335	688.1345
0.7500	2.1654	7.2877	32.7068	184.4511
1.0000	1.7539	4.7569	16.9468	74.7000
1.5000	1.3046	2.6443	6.9497	22.1980
2.0000	1.0554	1.7488	3.7467	9.6796
2.5000	0.8931	1.2668	2.3254	5.1316
3.0000	0.7776	0.9708	1.5728	3.0607
3.5000	0.6904	0.7732	1.1274	1.9753
4.0000	0.6219	0.6334	0.8428	1.3491
4.5000	0.5666	0.5301	0.6505	0.9616
5.0000	0.5207	0.4514	0.5149	0.7087

Table 4.4 : $\lambda = 3.0$

q	$E[Y]$	$E[Y^2]$	$E[Y^3]$	$E[Y^4]$
0.2500	4.9558	38.6687	396.5217	4852.3856
0.5000	3.0394	14.3198	91.1403	733.9918
0.7500	2.2667	7.8195	35.5877	202.2540
1.0000	1.8484	5.1716	18.7902	83.8583
1.5000	1.3885	2.9316	7.9302	25.8627
2.0000	1.1312	1.9665	4.3671	11.5986
2.5000	0.9625	1.4403	2.7561	6.2873
3.0000	0.8416	1.1138	1.8898	3.8193
3.5000	0.7499	0.8938	1.3703	2.5034
4.0000	0.6777	0.7369	1.0347	1.7328
4.5000	0.6190	0.6203	0.8056	1.2498
5.0000	0.5702	0.5308	0.6426	0.9309

5 The $100p$-percentage point of the four-parameter type I generalized half logistic distribution

The $100p$-percentage point is obtained by setting the cumulative probability distribution function to p. That is

$$F(y_{(p)}) = p \Rightarrow (\lambda + 1)^q(\lambda + e^{y_{(p)}})^{-q} = 1 - p. \tag{5.1}$$

Solving for $y_{(p)}$ gives

$$y_{(p)} = \ln[\frac{1 + \lambda(1 - \sqrt[q]{1-p})}{\sqrt[q]{1-p}}]. \tag{5.2}$$

This reduces to ln3 when $q = 1$, $\lambda = 1$ and $p = 0.5$ which corresponds to the value of median of the standard half logistic distribution. The knowledge of y_p for several values of p provides an indication of how the unit probability mass is distributed over the real line.

6 The mode of the four-parameter type I generalized half logistic distribution

The mode of a probability density function is obtained by equating the derivative of the density function to zero and solve for the variable. therefore,

$$f_Y(y; q, \lambda) = \frac{q(\lambda + 1)^q e^y}{(\lambda + e^y)^{q+1}}, \quad 0 \leq y < \infty, \quad q > 0, \lambda > 0.$$

$$f'_Y(y; q, \lambda) = q(\lambda + 1)^q[\frac{(\lambda - qe^y)e^y}{(\lambda + e^x)^{q+2}}]. \tag{6.1}$$

Equating the derivative to zero and solve gives $y = -\infty$ or $y = \ln(\lambda/q)$. Since $y = -\infty$ is not in the interval under consideration, the mode $= \ln(\lambda/q)$. This is the mode of the distribution as confirmed when $q = 1$ and $\lambda = 1$ which gives the mode of standard half logistic distribution as zero.

7 Theorems that characterize the four-parameter type I generalized half logistic distribution

Here we state and prove two theorems that characterize this distribution

Theorem 7.1: *The random variable $Y = \ln(2x - \lambda)$ is a generalized half logistic if and only if the random variable X follows a generalized Pareto distribution with parameter q and λ is a positive constant.*

Proof: If X follows a generalized Pareto distribution with parameters q, then

$$f_X(x;q) = \frac{q}{x^{q+1}}, \quad x > 1, \quad q > 0. \tag{7.1}$$

Since the random variable $y = \ln(2x - \lambda)$ implies that $x = \frac{\lambda + e^y}{2}$ and the Jacobian of the transformation is $|J| = \frac{1}{2}e^y$. Therefore

$$f_Y(y) = |dy/dx| f_X(y) = \frac{Ke^y}{(\lambda + e^y)^{q+1}},$$

where $k = q(\lambda + 1)^q$. Hence

$$f_Y(y;q,\lambda) = \frac{q(\lambda+1)^q e^y}{(\lambda + e^y)^{q+1}}, \quad y > 0, q > 0, \lambda > 0 \tag{7.2}$$

which is the probability density function of the four-parameter type I generalized half logistic distribution.

Conversely, if Y has a four-parameter type I generalized half logistic distribution with distribution function shown in equation (7.2), then the cumulative distribution function (cdf) of the random variable X is

$$F_X(x) = Pr[X \leq x] = Pr[\frac{\lambda + e^y}{2} \leq x] = F_Y[\ln(2x - \lambda)]. \tag{7.3}$$

By making use of the cdf of Y in equation (3.1)

$$F_X(x;q) = 1 - (\frac{\lambda+1}{2})^q x^{-q},$$

and

$$f_X(x;q) = (\frac{\lambda+1}{2})^q \frac{q}{x^{q+1}}.$$

By omitting the constant, the density of X can be written as

$$g(x) \propto \frac{q}{x^{q+1}}. \tag{7.4}$$

Since any density function proportional to the right hand side of equation (7.4) is that of a generalized Pareto random variable, the proof is complete.

Theorem 7.2: *The random variable Y is four-parameter type I generalized half logistic with parameters q and λ if and only if the density function f satisfies the homogeneous differential equation*

$$(\lambda + e^y)f' - (\lambda - qe^y)f = 0, \tag{7.5}$$

where prime denotes differentiation.

Proof: Suppose Y is four-parameter type I generalized half logistic with parameters q and λ with probability density function (pdf) shown in the equation (2.3). it is easily shown that the probability density function satisfies equation (7.5).

Conversely, we assume that f satisfies (7.5). Separating the variables in (7.5) and integrating, we have

$$\int \frac{f'}{f} dy = \int (\frac{\lambda - qe^y}{\lambda + e^y}) dy \tag{7.6}$$

$$\ln f = \lambda \int \frac{dy}{\lambda + e^y} - q \int \frac{e^y}{\lambda + e^y} dy$$

$$\ln f = \ln(\frac{}{\lambda + e^y}) - q\ln(\lambda + e^y) + \ln c$$

$$\ln f = \ln(\frac{e^y}{(\lambda + e^y)^{q+1}}) + \ln c. \tag{7.7}$$

Therefore,

$$f = \frac{Ce^y}{(\lambda + e^y)^{q+1}}, \quad 0 < y < \infty, \quad q > 0, \ \lambda > 0, \tag{7.8}$$

where C is a constant. The value of C that makes f a probability density function is $C = q(\lambda + 1)^q$. This completes the prove.

8 Conclusion

We have established the probability density function of a four-parameter type I generalized half logistic distribution with its cumulative distribution function, survival function and hazard function. The moments $E[Y^n]$ for $n = 1, 2, 3, 4$ have been tabulated for some values of parameters q and λ. The $100p$-percentage point and the mode of the distribution were presented. Two theorems that characterized the distribution were stated and proved. Further generalization of the distribution to contains more parameters is currently under study.

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