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A JOURNEY THROUGH THE BEAUTIFUL WORLD OF DIFFERENTIAL EQUATIONS

By

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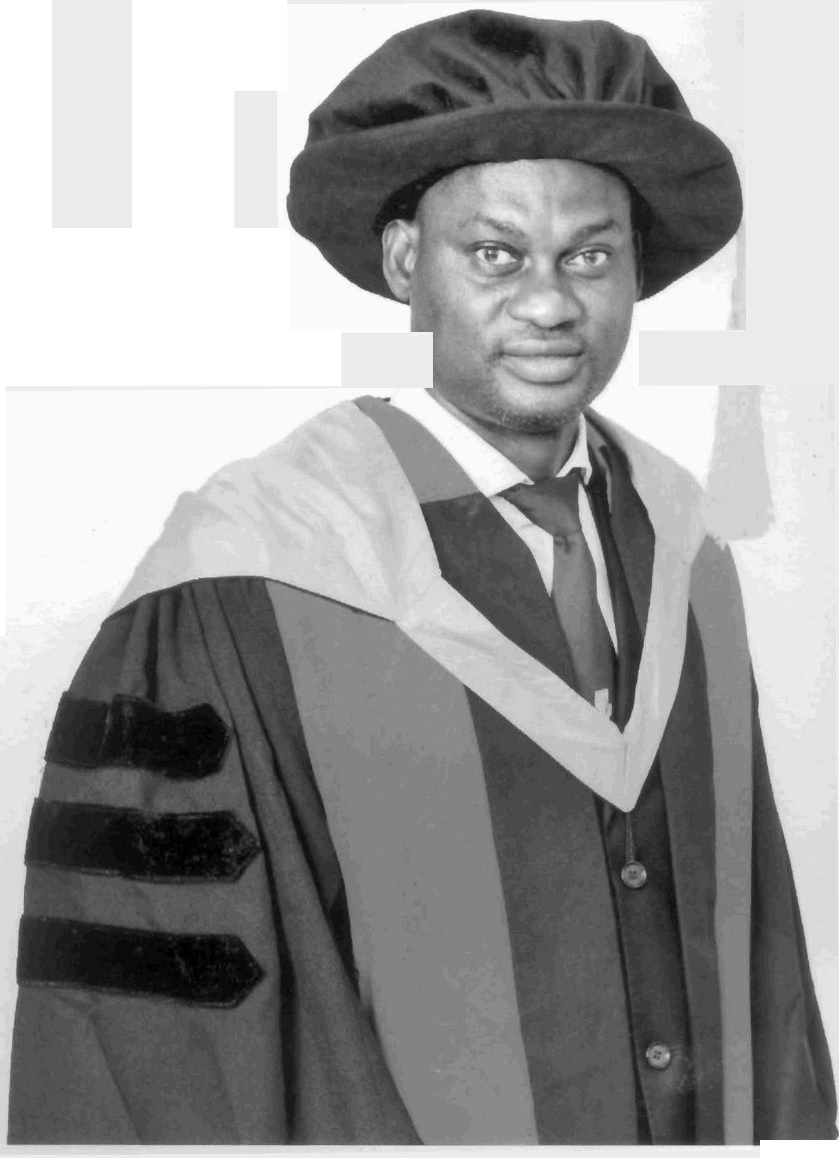

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An Inaugural Lecture Delivered at Oduduwa Hall, Obafemi Awolowo University, Ile-Ife on Tuesday 8th December, 2015

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As in the sea between Scylla and Charybdis the helmsman is ever in danger, yet he will be thought shrewd and sagacious, if, keeping his ship on a straight course between the two, avoiding the rocks on the one side and the maelstrom on the other, he brings his ship safely to harbour. So in learning, the scholar is tossed between difficulties and adversities; but he will be worthy of praise and glory, if, directing his mind and proper reason around them, likewise avoiding any impediment or contention, he penetrates without hindrance into the Truth he seeks. Carlo Vitali (1729).

1. Preambles

The Vice Chancellor, Sir, it gives me a great pleasure to stand before you today to deliver the 281st Inaugural Lecture of this University. I am humbled by your presence this evening. First and foremost, I would like to give thanks to the Almighty God through our Lord Jesus Christ for making this day a reality. He has been wonderful to me in the course of my academic carrier and without His grace and mercy, I would not be where I am today. Indeed, I have really enjoyed God's favour in my life and I appreciate Him even more for His kindness. I also appreciate the Obafemi Awolowo University for this opportunity as I join, on this podium, my two hundred and eighty worthy predecessors in the Inaugural Lecture Series at Ife.

Inaugural Lectures are known to be an important occasions in an academic's career. It provides newly appointed professors the opportunity to inform colleagues, the university community, the host community and the general public of their work to date, including current research and future plans. It further provides the opportunity for professors to announce their ultimate entry into the jealously guarded league of the most senior university teachers in the citadel of learning. Usually, Professors are expected to give their Inaugural Lectures within a specific period of their appointments or pronouncements as the case may be. Unlike the conventional lectures in the classrooms where the audience would have known the course contents; where the audience would have the opportunity of asking the lecturer questions; and where the audience would even have read ahead of the class, an Inaugural Lecture hardly give these opportunities beyond the announcements on the various channels where the University community would have had an idea of the topic of the lecture. In following this tradition, I wish to solemnly render before you and to the entire human race, now and in future, an account of my humble contribution to knowledge, especially within and from the standpoint of my own field of specialization, namely, mathematics, particularly in the sphere of differential equations.

It was a difficult task for me to choose a topic that would be appropriate for this Inaugural Lecture. I was very conscious of the fact that I would be speaking to a diverse audience in an environment where mathematics is yet to take its prime and appreciable position. Mathematicians are often seen here as a group of essentric, 'crazy' and 'mad' people. After a great deal of support from my wife, I reached a harmonizing act and found the current topic of "'A Journey through the Beautiful World of Differential Equations"' quite appropriate.

The Department of Mathematics of this University is a foundation department. It has produced till date a total of thirteen worthy Professors. I will like to acknowledge our respected Professors in the Department of Mathematics. I will also like to pay tribute to Professor B.L. Sharma who delivered his inaugural lecture titled "Mathematics through the Ages" and late Professor R.F. Abiodun who delivered his inaugural lecture titled "Mathematics: The Gate and Key of the Sciences"'on 8th December, 1987. In effect, the last Inaugural Lecture from the Department of Mathematics was given exactly twenty eight years ago, at this time and venue! It is therefore with a deep sense of humility that I inform this gathering that I am the third Professor from the Department of Mathematics in this University that will give an Inaugural Lecture and the first that would do so in the area of differential equations.

I have therefore set a very arduous task for myself in making sure that to the best of my abilities, that this audience is carried along, though in a manner that the lecture will not be over simplified. In order to achieve this, at times in the course of this lecture I will attempt to break things down to a rudimentary level when necessary and at other times, I shall not hesitate to operate with the seemingly esoteric mathematical tempo. However, I hope to be as non-technical as possible but I will have to use some mathematical language and notation and I apologize in advance and also my apologies to specialists to whom the matters explained here may rather sound pedantic.

Publish or Perish

The caption 'publish or perish' captured my attention the very first time I entered the office of Professor Anthony Uyi Afuwape. It was boldly written at one corner on the big blackboard in his office at the 'White House'. For a long time, I was wondering what this caption meant until I summoned enough courage to ask the respected professor what it was all about. He would take his time to explain to me comprehensively, what research was all about and in particular how to be a good researcher in mathematics. He then stood up gently from his chair to lecture me with matchless pedagogical finesse and exposed me to some trends as at then in differential equations especially on the qualitative behaviour of solutions via Lyapunov's second method and frequency domain techniques. Third order differential equations were his focal point. I kept this wonderful exposition intensely in my memory. I was fascinated not only by the beauty and the elegance of his discourse but also by the crystal clarity of his arguments and reasoning as well as the manner in which he made himself understood without any compromise to the perfect rigour of his exposition. He conceived his ideas in such a way that it created the impression that I was directly involved in the discovery of the considered notions, theorems and their proofs. I had earlier observed that our professor concentrated on third order equations in his research work and when I asked him for the reasons, he gave an answer that had since been our "little secret" and which I will not share with this audience. Nonetheless, he cleverly mandated me to work on fifth order nonlinear differential equations as there were very few contributions in this direction and as a matter of fact, there was no contribution at all through the frequency domain technique. This was the beginning of my journey into the beautiful world of differential equations. A journey that would expose me to this ever captivating and scintillating aspect of mathematics in whose amusing theorems lie the manifestation of its applications. It is indeed a journey whose end leads to another journey on and on and on ...

The journey with oasis in the desert of differential equations actually started in Kosy Bethel Primary School in Ilupeju, Lagos and would take me to Baptist Academy also in Lagos, for my secondary education. When it was time to gain admission into higher institution, I was privileged to have had three admission offers simultaneously. One, to study Chemistry at Obafemi Awolowo University, another, to study Mathematics at Ogun State University and the last, to study Civil Engineering at Yaba College of Technology. My inquisitiveness for solving problems and the very strong wish of my loving mother made me to embrace Mathematics. She was fond of calling me a 'Professor' right from an age that I did not understand what a secondary school was! Little did I know that my mother had a third eye as she was prophesying into my life. A critical moment would later occur in my life when I won the Honours Award for Outstanding Youth Corps Member in Ondo State. I was tossed between embracing automatic employment and enrolling for an M.Sc. Degree Programme at Obafemi Awolowo University on the one hand and on the other, jetting out for greener pastures to the United Kingdom. This was a defining moment! I opted for the former and was employed as a Graduate Assistant in October 1997. I defended my Ph.D. Thesis entitled 'Lyapunov and Frequency Domain Methods in Some Nonlinear Ordinary and Delay Differential Equations.' in 2006 and five years later, in 2011, I was promoted to the rank of Professor of Mathematics.

Mr Vice Chancellor, Sir, I shall now move from this segment of my academic Odyssey to the day's actual task.

2. Introduction

Since the beginning of time, human beings seem constrained to organise in such a way that they have a practical need to count certain things like crops, animals, and so on. In those days, fingers and thumbs provide tools for counting. Moreover, whenever any sort of record is needed, notches in a stick, a bone or a stone are the natural solutions. From relevant history, the Palaeolithic people in Central Europe and France recorded numbers on bones around 30000 BC. Documentations on early development of Mathematics though somewhat speculative, pieced together from archaeological rubbles, architectural relics, educated speculations and others studying early societies, clearly reveal that when agriculture was conceived around 15,000 B.C., human beings had to address the notion of multiplicity and space (when counting animals or distributing crops). Over the centuries, generations of scholars have refined and stretched these concepts into the evolution of Arithmetic and later into Algebra. Furthermore, the need to solve problems of providing shelter and dealing with areas of fields and pastures led to simple geometrical situations. Consequently, from the early stages of civilization, the initial history of Mathematics was that of Geometry and Algebra.

Further developments of Mathematics in the early years as reported in Dunham (1990) came along with the civilization of ancient Egypt where records indicate that by 2000 B.C., the Egyptians had a primitive numeral system as well as some geometric ideas about triangles, pyramids, and the like. Algebra was developed only far enough to calculate the volume of a pyramid. Between 1900 and 1600 B.C., Mathematics advanced significantly in Mesopotamia with the Babylonians solving fairly sophisticated problems with a definite algebraic character. Sixty was the number system base-a system that we have inherited and preserve to this day in our measurement of time (60 seconds per minute) and angles ($6 \times 60 = 360$ in a circle). While it is obvious that the base 10 we now embrace was derived from the 10 fingers of human hands, the Babylonians' base 60 certainly did not emanate from sixty fingers of human hands!

The intellectual exploits of the Greeks especially, the epic poet Homer, the historians Herodotus and Thucydides, the dramatists Aeschylus, Sophocles, and Euripides, the politician Pericles, and the philosopher Socrates influenced the great Thales (640-546 B.C.), one of the so-called "Seven Wise Men" of antiquity. Thales of Miletus was known as the father of demonstrative Mathematics and as such, he was the earliest known mathematician. That was the era that started the use of 'Q.E.D' upon the completion of a proof, which abbreviates the Latin "Quod erat demonstrandum"- meaning "which was to be proved". After Thales, then came Pythagoras who founded a scholarly society now known as the Pythagorean brotherhood. According to Dunham (1990), in the contemplation of the world about them, the Pythagoreans recognized the special role of "whole number" as the critical foundation of all natural phenomena. Whether in music, or astronomy, or philosophy, the central position of "number" was evident everywhere. The modern notion that the physical world can be understood by "mathematisation" owes more than a little to this Pythagorean viewpoint. In the world of Mathematics proper, the Pythagoreans gave us two great discoveries. One, of course, was the incomparable Pythagorean theorem. As with all other results from this distant time period, we have no record of the original proof, although the ancients were unanimous in attributing it to Pythagoras. In fact, legend has it that a grateful Pythagoras sacrificed an Ox to the gods to celebrate the joy his proof brought to all concerned (except, presumably, the Ox). The other significant contribution of the Pythagoreans was received with considerably less enthusiasm, for not only did it defy intuition, but it also struck a blow against the pervasive supremacy of the whole number. In modern parlance, they discovered irrational quantities!

Further contributions were made about 440 B.C. by Hippocrates who is remembered for his composition of the first Elements, that is, the first exposition developing the theorems of geometry precisely and logically from a few given axioms or postulates. Then came the era of Plato (427-347 B.C.) and Eudoxus (408-355 B.C.). Dunham (1990) noted that of the many subjects studied at the Plato's Academy, none was more highly regarded than Mathematics. The subject certainly appealed to

Plato's sense of beauty and order and represented an abstract, ideal world unsullied by the humdrum demands of day-to-day existence. Moreover, Plato considered Mathematics to be the perfect training for the mind, its logical rigour demanding the ultimate in concentration, cleverness, and care. Legend has it that across the arched entryway to his prestigious Academy were the words "Let no man ignorant of geometry enter here." Explicit sexism notwithstanding, this motto reflected the view that only those who had first demonstrated a mathematical maturity were capable of facing the intellectual challenge of the Academy. We might say that Plato regarded geometry as the ideal entrance requirement, the Scholastic Aptitude Test (SAT) of his day.

More advancements to Mathematics were made by other Greeks and the Romans Scholars until the middle ages in 476 A.D., when the Roman Empire came to an end. While the West was almost grounded, a new era of contributions from Arabian, Chinese and Indian Mathematicians flourished. For instance, our modern base ten number system featuring zero as a place holder was developed in the Eighth Century in India. The basis for algebra was developed in the Arabic World in the Eighth and Ninth Centuries. In fact, about 810 A.D. Al-Khwarizmi wrote important works on Hisab al-jabr w'al-muqabala (Calculation by Completion and Balancing) that gives us the word algebra from "al-jabr" which refers to a quantity from one side of an equation to the other. From al-Khwarizmi's name, as a consequence of his arithmetic book, comes the word "algorithm".

Around 1500 A.D. the views of intellectualism in Europe started changing which signaled the end of the Middle Ages and heralded the Modern World which would ever since witness the creation of powerful, new and insightful Mathematics.

Many outstanding contributions have been made from great minds from then till today with the emergence of what we recognize as Mathematics in the modern world as opposed to the geometrical discussions of ancient times. From the work of Girolamo Cardano in 1545 which gave for the first time a general solution of the cubic equation, and some special cases of the quartic equation to the expositions of Kepler (Nova stereometria doliorum vinarorum (Solid Geometry of a Wine Barrel), an investigation of the capacity of casks, surface areas, and conic sections), Galileo (Sidereus Nuncius (Message from the stars) which describes his astronomical discoveries), Descartes (La Gomtrie which describes his application of algebra to geometry), Newton (discoveries of the binomial theorem and differential calculus, The Principia or Philosophiae naturalis principia mathematica (The Mathematical Principles of Natural Philosophy)), Leibniz (detailed his differential calculus in Nova Methodus pro Maximis et Minimis, itemque Tangentibus) and John Napier (Introduction of Logarithms). Fermat (proposition of Fermat's last theorem), Blaise Pascal (Essay pour les coniques (Essay on Conic Sections) and Treatise on the Equilibrium of Liquids on hydrostatics) in the Seventeenth Century. With calculus at its center, an ever widening body of knowledge began to take shape. Notable mathematicians that would later prove their worths include Augustin-Louis Cauchy (1789-1857) who gave a formal definition for the limit equivalent to the modern $\epsilon - \delta$-definition; Evariste Galois (1811-1832), Niels Abel (1802-1829) and Karl Frederic Gauss (1777-1854).

Mr Vice Chancellor, Sir, let us fast track to the era of differential equations.

2.1. Differential Equations: Origin, Developments and Roles in the Changing World

The study of differential equations started with the efforts of Gottfried Wilhelm von Leibniz in 1675, when he came up with the equation:

$$\int x dx = \frac{1}{2}x^2$$

Isaac Newton began the search for general methods of integrating differential equations when he classified first order differential equations into following:

$$\frac{dy}{dx} = f(x);$$

$$\frac{dy}{dx} = f(x, y);$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = u.$$

Newton classified the first two equations as ordinary differential equations since they contain only ordinary derivatives of one or more dependents as ordinary differential equations since they contain only ordinary derivatives of one or more dependent variables, with respect to a single independent variable. The third class involved the partial derivatives of one dependent variable and today are called partial differential equations. Other earlier contributors include James Bernoulli (1654-1705) and his brother John Bernoulli (1667-1748) who worked on the problems of finding a curve along which a body will fall with uniform vertical velocity (Isochrone) and finding the equation of the path down which a particle will fall from one point to another in the shortest time (Brachistochrone) respectively. Leonhard Euler (1707-1783) reduced second order equations to first order by finding an integrating factor. Joseph Lagrange (1736-1813) conceptualised adjoint of a differential equation. Jean d'Alembert (1717-1783) worked on linear equation with constant coefficients and derived conditions under which the order of a linear differential equation could be lowered.

Due to the difficulty in finding integrating factors, the formal methods became insufficient. Solutions with special properties were required, and thus, criteria guaranteeing the existence and uniqueness of such solutions became important which metamorphosed into qualitative behaviour of solutions. We can learn so much about the solutions to a differential equation without solving the equation. For instance, consider the equation $x^{'} = f(x)$, we can find equilibrium points by finding zeros $\bar{x}$ of f. We can often determine their stability by examining the eigenvalues of the derivatives; when this fails, we can still often determine stability by looking at the effects of nonlinear terms. We can ask other qualitative questions about the solutions. For example, are there periodic solutions? Are they stable? How does the system respond to parameter changes. The first person to carry out a major investigation along these lines was Henry Poincaré, with a new epoch in the development of the theory of differential equations called "qualitative theory of differential equations". This qualitative theory is now the most actively developing area of the theory of differential equations, having the most important applications in diverse areas as engineering, economics, physical and biological sciences etc.

It is well known that mathematical formulations of many physical problems often result in differential equations that are nonlinear. However, in many cases, it is possible to replace a nonlinear differential equation with a related linear differential equation that approximates the actual nonlinear equation closely enough to give useful results. Often, such linearisation is not possible or feasible; when it is not, the original nonlinear equation itself must be tackled. Whereas the general theory and methods of dealing with linear differential equations constitute a highly developed branch of Mathematics, very little of a general nature is known about nonlinear differential equations. By nonlinear differential equations, we mean equations in which the unknown function and/or its higher derivatives (with their powers) occur strongly coupled in at least one of the terms of the expressions. Generally, the study of nonlinear differential equations is confined to a variety of very special cases and the method of solution usually involves one or more of a limited number of different methods. There are several important differences between linear differential equations and nonlinear differential equations: in the case of linear ordinary differential equations, it is often possible to derive closed-form expressions for the solutions of the equations. In general, this is not possible in the case of nonlinear ordinary differential equations. As a consequence, it is desirable to be able to make some predictions about the behaviour (qualitative analysis) of nonlinear ordinary differential equations even in the absence of closed-form expressions for the solutions of the equations. The analysis of nonlinear differential equations makes use of a wide variety of approaches and mathematical tools than does the analysis of linear differential equations. The main reason for this variety is that no tool or methodology in nonlinear differential equations analysis is universally applicable in a fruitful manner.

Some Basic Concepts of Differential Equations

Definition 2.1.1.

A differential equation is an equation that involves one or more derivatives of some unknown function or functions.

Definition 2.1.2.

An ordinary differential equation is an equation involving one independent variable; one or more dependent variables, each of which is a function of the independent variable; and ordinary derivatives of one or more of the independent variables, i.e.

$$y' = \frac{dy}{dx} = 2x^2 + 3x + 5.$$

Definition 2.1.3.

The order of an ordinary differential equation is said to be n if the order of the highest derivative appearing in the equation is n, i.e.

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 2y = 0$$

is a second order differential equation.

Definition 2.1.4.

A solution of an ordinary differential equation with one dependent variable is a differentiable function of the independent variable that satisfies the equation on some interval. In other words, if we substitute the function for the dependent variable, we obtain a result that is valid on the interval.

Definition 2.1.5.

A partial differential equation is an equation containing more than one independent variable, one or more dependent variables, and partial derivatives of one or more of these dependent variables.

Definition 2.1.6

A delay differential equation is an equation for a function of a single variable, usually called time, in which the derivative of the function at a certain time is given in terms of the values of the function at earlier times.

Definition 2.1.7.

A stochastic differential equation is an equation in which the unknown quantity is a stochastic process and the equation involves some known stochastic processes, for example, the Wiener process in the case of diffusion equations.

Definition 2.1.8.

A differential algebraic equation is a differential equation comprising differential and algebraic terms, given in an implicit form.

Definition 2.1.9.

A **general solution** of an *nth*-order differential equation is a solution containing n essential arbitrary constants. A **particular solution** of a differential equation is any one solution. It is usually obtained by assigning specific values to the constants in the general solution. A particular solution can be represented as a curve in the xy-plane called an integral curve. The general solution of a differential equation is the set of all integral curves, or the family of all integral curves. A solution of a differential equation that cannot be obtained from a general solution by assigning particular values to the arbitrary constants is called a **singular solution**.

There are several other important basic concepts but for the purpose of this inaugural lecture, we shall only limit ourselves to the above defined concepts. We shall now state some of the areas where differential equations can be applied.

These include but are not limited to Sound waves in air; modeling cancer growth or the spread of disease in medicine; optimum investment strategies in economics; linearized supersonic airflow; crystal growth; cryocooler modeling; casting of materials; materials science; electromagnetics analysis for detection by radar; material constitutive modeling and equation of state; underwater acoustic signal processing; reentry simulations for the Space Shuttle; rocket launch trajectory analysis; trajectory prescribed path control and optimal control problems; design and analysis of control systems for aircraft; molecular and cellular mechanisms of toxicity; transport and disposition of chemicals through the body; immuno-assay chemistry for developing new blood tests; radio interferometry; free mesons in nuclear physics and seismic wave propagation in the earth (earthquakes).

Let us shed more light on further applications of differential equations.

Modeling of DNA denaturation-DNA, the short name for deoxyribonucleic acid, is a nucleic acid that contains the genetic instructions used in the development and functioning of all known living organisms. Chemically, a DNA consists of two long polymers of simple units called nucleotides, with backbones made of sugars and phosphate groups. These two strands run in parallel and form a double helix. Attached to each sugar is one of four types of nucleotide molecules, also called bases, named by letters A (adenine), C (cytosine), G (guanine), T (thymine), so that only A and T, C and G, from opposite strands may bind to form pairs. Differential equations are useful in studying the dynamics of DNA and its function through mathematical modeling.

Carbon Dating-The carbon dating of paintings and other materials such as fossils and rocks lies in the phenomenon of radioactivity discovered at the turn of the century. It can be shown that the radioactivity of a substance is directly proportional to the number of atoms of the substance present. Thus, if $N(t)$ denotes the number of atoms present at time t, then $\frac{dN}{dt}$, the number of atoms that disintegrate per unit time, is proportional to N; i.e., $\frac{dN}{dt} = -\lambda N$.

Forensic Analysis in Homicide-The time of death of a murdered person can be determined with the help of modeling through differential equation. For instance, if the body of a dead person presumably murdered and the problem is to estimate the time of death. For the fact that the body will radiate heat energy into the room at a rate proportional to the difference in temperature between the body and the room, the time of death can be estimated from body's current temperature and knowing how long it would have had to lose heat to reach this point.

Drug Concentration in Human Body-To combat the infection to a human body, appropriate dose of medicine is essential. Because the amount of the drug in the human body decreases with time, medicine must be given in multiple doses. The rate at which the level y of the drug in a patient's blood decays can be modeled by the decay equation.

Predator-prey equations-The predator prey equations are a pair of first order, nonlinear differential equations frequently used to describe the dynamics of biological systems in which two species interact, one as a predator and the other as prey.

Peter Deuflhard in one of his CIME Lectures highlighted the following as some of the recent trends in the application of differential equations:

Pharmaceutical Drug Design-The design of highly specific drugs on the computer, the so-called rational drug design (as opposed to irrational drug consumption), is a fairly recent dream of biochemistry and pharmaceutical industry. At first mathematical glance, drug design seems to involve the numerical integration of the Hamiltonian differential equations that describe the dynamics of the molecular system under consideration.

Countable Ordinary Differential Equations in Polymer Industry-Chemically speaking, polymers are a special kind of macromolecules: chains of simple molecules or molecular groups, the monomers. The chains typically consist of ten thousand up to ten million monomers, say, and may be linear or even bifurcating. Copolymers are built from more than one type of monomer. Polymer materials include a variety of synthetic materials such as lacquers, adhesives, PVC, and MMA. Mathematically speaking, models of polyreaction kinetics involve a huge set of ordinary differential equations, usually nonlinear and stiff. The numbers of ordinary differential equations again range from ten thousand up to ten million in this case.

Biopolymerization- This problem deals with an attempt to recycle waste of synthetic materials in an ecologically satisfactory way which is certainly an important problem of modern industrial societies. An attractive idea in this context is to look out for synthetic materials that are both produced and eaten by bacteria under different environmental conditions, of course. The macromolecular reaction steps of production and degradation of PHB can be summarized in the mathematical model.

Cancer Therapy Planning-Partial differential equation models arising in medicine (example: cancer therapy hyperthermia) and high frequency electrical engineering (example: radio wave absorption). In this type of application the 3D geometry say, of human patients motivates the choice of tetrahedral finite element methods (FEM). Reliability plays the dominant role in medicine, which is a nice parallelism with the intentions of Mathematics. Numerical speed is required to permit a fast simulation of different scenarios for different patients. In other words: the situation both requires and deserves the construction of highly efficient algorithms, numerical software and visualization tools.

Electrodynamics-Maxwell's equations together with the Lorentz force law form the foundation of classical electrodynamics, classical optics, and electric circuits which gave birth to modern electrical and communications technologies.

3. Differential Equations, Love Affairs and Poetry

Love is one of the most important problems in human lives and we can learn so much from the dynamics of love with the aid of differential equations. The first direct work in this direction was given by Strogatz (1998) who reports in his one-page paper, his success in teaching harmonic oscillators by making reference to Romeo and Juliet. The story of Strogatz can be modified as: Juliet is in love with Romeo but Romeo is a fickle lover. The more Juliet loves him; the more he begins to dislike her. But when she loses interest, his feelings for her warm up. She, on the other hand, tends to echo him. Her love grows when he loves her, and it turns to hate when he hates her.

A simple model for their ill-fated romance is

$$\begin{aligned} \dot{X}_1(t) &= -\theta_1 X_2(t) \\ \dot{X}_2(t) &= \theta_2 X_1(t), \end{aligned} \tag{3.1}$$

where $X_1(t)=$ Romeo's love/hate for Juliet at time t, $X_2(t)=$ Juliet's love/hate for Romeo at time t, and $\theta_1 > 0$ and $\theta_2 > 0$ are intensity constants. The governing equations of the model are those of a simple harmonic oscillator- a never ending cycle of love and hate. Thus their affair ended on a sad note.

A more robust model based on differential equations that describe the dynamics of love between two persons by accounting for their personalities was given by Rinaldi (1998) [stimulated by the work of Jones (1995) on Petrarch's Canzoniere]. According to Rinaldi, the model makes specific reference to a very special and famous case of unrequited love. A detailed linguistic and stylistic analysis of all the dated poems addressed by Petrarch to his platonic mistress Laura has allowed Jones to conjecture that the poet's emotions follow for approximately 20 years a quite regular cyclical pattern, ranging from the extremes of ecstasy to despair. Let us now look at the interplay among differential equations, love affair and poetry.

We will consider Rinaldi's account: the emotions of Laura and Petrarch are modeled by means of three ordinary differential equations where Laura is described by a single variable $L(t)$, representing her love for the poet at time t. Positive and high values of L mean warm friendship, while negative values are associated with coldness and antagonism. The personality of Petrarch is more complex; its description requires two variables: $P(t)$, love for Laura, and $Z(t)$, poetic inspiration. High values of P indicate ecstatic love, while negative values stand for despair.

$$
\begin{aligned}
\frac{dL(t)}{dt} &= -\alpha_1 L(t) + R_L(P(t)) + \beta_1 A_P, \\
\frac{dP(t)}{dt} &= -\alpha_2 P(t) + R_P(L(t)) + \beta_2 \frac{A_P}{1+\delta Z(t)}, \\
\frac{dZ(t)}{dt} &= -\alpha_3 Z(t) + \beta_3 P(t),
\end{aligned}
\tag{3.2}
$$

where $RL(.)$ and $RP(.)$ are reaction functions. $A_P[A_L]$ is the appeal (physical, as well as social and intellectual) of Petrarch [Laura], and all Greek letters are positive constant parameters (this means that variations in the personalities of Laura and Petrarch due to aging or other external factors are not considered). The rate of change of the love of Laura (the first equation in (3.2)) is the sum of three terms. The first, negative for positive L, describes the forgetting process characterizing each individual. The second, namely $R_L(P)$, is the reaction of Laura to the love of Petrarch, while the third is her response to his appeal. The second equation in (3.2) is similar to first with one relevant exception: the response of Petrarch to the appeal of Laura depends also upon his inspiration Z. This takes into account the well-established fact that high moral tensions, like those associated with artistic inspiration, attenuate the role of the most basic instincts. And there is no doubt that the tensions between Petrarch and Laura are of a passionate nature. For example, the poet says

Con lei foss'io da che si parte il sole,
et non ci vedess' altri che le stelle,
sol una nocte, et mai non fosse l'alba;

[Would I were with her when first sets the sun,

and no one else could see us but the stars,
one night alone, and it were never dawn].

In his Posteritati he confesses "Libidem me prorsus expertem dicere posse optarem quidem, sed si dicat mentiar" [I would truly like to say absolutely that I was without libidinousness, but if I said so I would be lying]. Finally, the third equation in (3.2) simply says that the love of Petrarch sustains his inspiration which, otherwise, would exponentially decay with a time constant $\frac{1}{\alpha_3}$. In other words, poetic inspiration is an exponentially weighted integral of the passion of the poet for his lover.

Rinaldi (1998) further asserts that a linear reaction function is not appropriate for Laura. In fact, only close to the origin can $R_L(P)$ be assumed to be linear, thus interpreting the natural inclination of a beautiful high-society lady to stimulate harmless flirtations. But Laura never goes too far beyond gestures of pure courtesy: she smiles and glances. However, when Petrarch becomes more demanding and puts pressure on her, even indirectly when his poems are sung in public, she reacts very promptly and rebuffs him, as described explicitly in a number of poems, like:

Mille fiate, o dolce mia guerrera,
per aver co'begli occhi vostri pace,
v'aggio proferto il cor; m voi non piace
mirar s basso colla mente altera.

[A thousand times, o my sweet enemy,
to come to terms with your enchanting eyes
I've offered you my heart, yet you despise
aiming so low with mind both proud and free].

The above poem, relative to the model can be interpreted mathematically as the use of a reaction function $R_L(P)$ which, for $P > 0$, first increases and then decreases. But the behavior of Laura is also nonlinear for negative values of P. In fact, when $P << 0$, i.e., when the poet despairs, Laura feels very sorry for him. Following her genuine Catholic ethic she arrives at the point of overcoming her antagonism by strong feelings of pity, thus reversing her reaction to the passion of the poet. This behavioral characteristic of Laura is repeatedly described by the poet:

Volgendo gli occhi al mio novo colore,
che fa di morte rimembrar la gente,
piet vimosse;onde,benignamente
salutando, teneste in vita il core.

[Casting your eyes upon my pallor new,
which thoughts of death recalls to all mankind,
pity in you I've stirred; whence, by your kind
greetings, my heart to life's kept true].

Again, the above poem is equivalent mathematically to saying that the function $R_L(P)$, besides having a positive maximum for $P > 0$, has a negative minimum for $P < 0$.

The above work of Rinaldi (1998) is yet another beauty of differential equations. It will be interesting to have a model of love dynamics between one man and many wives, between one woman and many husbands and any other combinations that I would not mention here since same sex marriage is a taboo in our Nigeria.

4. Differential Equations and Corruption

It is well known that corruption is a multifaceted social, political, and economic occurrence that affects all societies at different degrees - a substructure of an unavoidable part of human social interaction, prevalent in every society at any time since the very beginning of human history till today. It is a deviation from fair play interaction in the development of social relations which depends on the cultural context of a given society. It can also be viewed as a misuse of public power to gain profit in a more or less illegal way. In any case, corruption has many different faces in its concrete appearance and no single definition will be able to describe the whole picture in an adequate way. In Nigeria, the phenomenon of corruption has adverse impacts on the economic development and breeds negative work ethics with its attendant consequence of wasting and mis-allocating resources and thus leading to unequal distribution of income and wealth. Indeed, President Muhammadu Buhari of the Federal Republic of Nigeria on his part, defined corruption as the greatest form of human right violation.

Successive governments in Nigeria have been paying lip service in their fights against corruption. Though there are several literature on how to tackle corruption in Nigeria from the political, social and economical angles, but the problem still remains intractable. Unfortunately too, the attitude of our government towards funding of meaningful research in our universities to look at the problem comprehensively and proffer long lasting solutions to it has been abysmal. We must not deceive ourselves that we are solving corruption problems when we do not even understand the dynamics of corruption mathematically! Mathematics can help our society to understand the dynamics of this endemic corruption that has brought the country to its knees. Through differential equations, we can study such dynamics of corruption as long as we can clearly define the aim and limitations of the adopted approach.

According to Hathroubi and Trabelsi (2014), in a given country, the total population (P_t) can be decomposed, at every moment, in three categories:

$$P_t = SC_t + C_t + H_t,$$

where SC_t: is the fraction of the population susceptible to become corrupted (corruptible). It is non-corrupt persons who may become corrupted if they are in contact with individuals initially corrupted (C_t);
H_t: is the fraction of the population immunised against corruption. These are honest individuals who do not change their attitude regardless of the situation ($\bar{H}$). There are also people who were corrupt and were caught and pay for their mistakes by imprisonment or if they are removed from work to retirement ($\frac{dH_t}{dt}$), then $H_t = \bar{H} + \frac{dH_t}{dt}$.
The model is also based on the following assumptions:
H1: that a corruptible transmits corruption immediately when he/she becomes corrupted;
H2: that corrupt-corruptible exchange is proportional to the statistical average of contacts. The coefficient of proportionality is by definition, the rate of corruptibility (i);
H3: that corrupt-honest relationship is proportional to the corrupt population. The coefficient of proportionality defines the "good repute rate or honesty rate" (equivalent to the cure rate

in an infected population usually known as the removal rate) (g). The spread of corruption is similar to an infectious disease and can be represented by a dynamic system:

$$
\begin{aligned}
\frac{dSC_t}{dt} &= -iSC_t.C_t, \\
\frac{dC_t}{dt} &= iSC_t.C_t - gC_t, \\
\frac{dH_t}{dt} &= gC_t,
\end{aligned}
\tag{4.1}
$$

with initial conditions for $t = 0$, $C_0 > 0$, $SC_0 > 0$ and $H_0 = \bar{H}$.

The above model can be used to study corruption as an epidemic phenomenon using an epidemic diffusion model. The dynamics of corruption and its impact on the composition of the population at a given time can also be determined alongside a threshold epidemiological corruption based on the approximation of the honest population. We can go further by asking the question - what can be done to slow down the infection, propagation or prevalence of corruption.

Mr Vice Chancellor, Sir, while I am not claiming that we can use this approach to eradicate corruption completely, I am of the opinion that if our government can adopt this mathematical approach in the fight against corruption, then we may heave a sigh of relief.

5. Differential Equations and Terrorism

Terrorism can be seen from the perspective of Mathematics, though there is abundant literature that discuss and analyze terrorist groups and terrorist activities. Similarly, the literature on civil war studies and rebel groups that militarily engage a state for control of part or all of an existing state. Ostensibly, terrorist groups are defined by their tactics of choice, violence that is targeted towards non combatants, intended for a much wider audience, and meant to help achieve political goals. The roots of terrorism can be identified as lying in economic, religious, psychological, philosophical, and political aspects of society. Incidents of terrorism can often be sparked by the deterioration of some local conditions, in a spatial sense, as perceived by a small segment of society. Nigeria is not immuned to all these as the North Eastern part of the country is bedeviled with the scourge of Boko Haram insurgency. All efforts to bring the activities of this group down have so far not yielded the expected result. The recent call by former President Olusegun Obasanjo challenging Nigerian universities to come up with solution to terrorism in the nation is a welcome development. Differential equations as a tool can make us to understand the dynamics of terrorism in Nigeria as well as in other countries inflicted with it. Let us take a look at terrorism via a mathematical model.

In an interesting article, Udwadia *et. al* (2006) presents a simple dynamical model of terrorism in terms of the dynamics of the population of individuals who engage in terrorist activities. The population is divided into two categories: terrorists (T) and nonterrorists (NT). The nonterrorists are further divided into those that are susceptible to terrorist propaganda, this segment of the population we called susceptibles (S) and those that are not susceptible to such propaganda are referred to as nonsusceptibles (NS). The number of terrorists (T) in a certain geographical region at time τ is $x(\tau)$ and population of susceptible (S), $y(\tau)$, and of nonsusceptibles (NS), $z(\tau)$.

According to Udwadia *et. al* (2006), the number of terrorists in a given period of time can change because of several reasons:
(1) direct recruitment by the terrorists of individuals from the susceptible population; the effectiveness of this is taken to be proportional to the product of the number of terrorists and the number of susceptibles;
(2) effect of anti terrorist measures that are directed directly at reducing the terrorist population, such as military and police action/intervention, which we assume increases rapidly with, and as the square of, the number of terrorists in the region under concern;
(3) number of terrorists that die from natural causes, or are killed in action, and/or self-destruct (as in the case of suicide bombers), which we assume to be proportional to the terrorist population itself;
(4) increase in the terrorist population primarily through the appeals by terrorists to other terrorist groups, through global propaganda using news media, and/or through the organized or voluntary recruitment/movement of terrorists from other regions into the region of concern, and also through population growth in this section of the population; this brings about an

increase in the terrorist population that we assume is proportional to the number of terrorists. We capture these four effects then through the following differential equation that we posit for the evolution of the terrorist population in the geographical region of concern;
(5) the increase in the susceptible population proportional to its own size.
The evolution of the susceptible population adduced from these effects (with other effects) can be expressed by a nonlinear system of three differential equations containing a total of 8 constant parameters

$$
\begin{aligned}
\frac{dx}{dt} &= axy - bx^2 + (c-1)x, \\
\frac{dy}{d\tau} &= -axy - ex^2y + fx + gy, \\
\frac{dz}{d\tau} &= ex^2y - f_1x + hz,
\end{aligned} \tag{5.1}
$$

where all the parameters are as defined in Udwadia *et. al* (2006).

There are many outstanding mathematicians in Nigeria that can help the government significantly in the fight against Boko Haram by coming up with a reasonable model for the dynamics which will point us in the right direction for asking the proper questions in order to predict and interdict terrorist activity.

6. Mathematics, Yoruba Culture and Ori Olokun

According to popular Yoruba myth of creation, Ile-Ife was where the world began. The kings, queens, and deities whose stories animate Yoruba history and art all trace their origins to this ancient centre of art and culture, today called Ife-Ife, which is now a bustling metropolis. To many Yoruba people and their descendants, who number more than 35 million and live in Nigeria, other areas of West Africa, and in the Western Diaspora, Ile-Ife is associated with the ancestors, with the first dynasties and deities, and with the invention of technology and art (Drewal and Schildkrout (2009)). As the ancestral home of some sixteen legendary kingdoms that flourished starting in about 1100-1200 C.E., Ile-Ife retains a spiritual primacy for Yoruba speaking people in Nigeria and beyond. The art of ancient Ife, though in many ways shrouded in mystery, is part of this cultural tradition; and it is mostly with reference to its art, rather than to written histories or comprehensive archaeology, that we are able to interpret Ile-Ife.

The ancient Yorubas operated an elaborate vigesimal (base-20) numeral system which makes use of addition, subtraction and multiplication. In this base-20 notational system for representing real numbers, the digits represent numbers using vigesimal notation which is a set of well-defined rules for representing quantities and operations with symbols. They also developed Ifa Oracle divination based on $16^2 = 16 \times 16 = 2^8$ corresponding to the vertices of an 8-dimensional hypercube and to the binary 2-choice Clifford Algebra Cl(8) which is the algebra of 16×16 real matrices. The Ifa Oracle has $N = 8$ ternary 3-structure as well as binary 2-structure. Ifa binary 2-structure and 3-structure correspond respectively to static states and to dynamic states.

Clifford Algebras (connected with the theory of quadratic forms and orthogonal transformations) are a type of associative algebra which generalise the real numbers, complex numbers, quaternions and several other hypercomplex number systems and have applications in digital image processing. Whereas Clifford Algebras were not known to European Mathematicians until the 19th century, its structures were known to the ancient Yorubas at least seven centuries earlier.

It is a known fact that human history is accompanied and punctuated by technological innovations. Generally speaking, artefacts and their use can be considered characteristic of human activity and their contribution at the cognitive level is largely acknowledged. Consciously or unconsciously, professional artists use mathematical processes all the time. According to Rutherford and Ahlgren (1990), there are at least three phases common to both the math and art process:

1. Representation of some aspects of things abstractly.
2. Manipulation of the abstractions by rules of logic to find new relationships between them.
3. Discovery of whether the new relationships say something useful about the original things.

The *Ori Olokun*, made of brass, 35 cm high, dated approximately 1200-1300 C.E., is currently resident in The British Museum, London, with Identification Af1939,34.1. It has characteristics of (1)-(3) above. It consists of two main parts: a circlet suggesting different layers of beads and a crest surmounting the forehead. This construction, sometimes called the 'phallic crest', is decorated in different ways. A plaited element, apparently corresponding to the tufts of hair worn in the present-day crowns, appears to be attached to the main structure with a kind of rosette consisting of various disc. The height has multiple layers of tube-shaped beads apparently applied either to an underlying construction or fixed -on hairs pulled through the beads. The forehead of Oriolokun is framed by an arc. The symmetry of lines on the face of *Ori Olokun* has mathematical interpretations. Knowingly or unknowingly, the ancient Yorubas had the knowledge of Mathematics.

FIGURE 1. *Ori Olokun*

7. Lyapunov and Frequency Domain Methods

The recent years have been marked by an ever-growing interest in the research of qualitative behaviour of solutions to nonlinear differential equations of higher orders. During these periods, new methods and outstanding results appeared. These were extensively summarized in the relevant literature. The major directions which must be emphasized in this context, consist in the investigation of solutions of nonlinear differential equations involving boundedness, convergence, stability, periodicity and almost periodicity of solutions. Some of the outstanding techniques used in investigating these qualitative properties of solutions include, Lyapunov's second method which involves constructing a suitable positive definite Lyapunov function whose derivative is negative definite. Another is topological degree method which demands the verification of continuity properties of a certain operator and the proof of existence of a particular a-priori bound. We can also mention the frequency domain method, which consists in the study of position of the characteristic polynomial roots in the complex plane. Each of these methods has its limitations, for instance, while there are no unique ways of constructing Lyapunov's functions; the frequency domain method, although overcomes the problems of constructing Lyapunov's functions, is narrower in scope than the Lyapunov's second method. The topological degree methods on the other hand are mainly used in proving existence of periodic and periodic like solutions.

The following are definitions of some basic notions.

Definition 7.1.1

Let

$$x'(t) = f(t, x(t)). \tag{7.1}$$

The solution $x(t)$ of equation (7.1) is said to be periodic if $x(t) = x(t+T)$ for $T > 0$, $-\infty < t < \infty$ for all t. T is called the period of x.

Definition 7.1.2

The function $f(t, x(t))$ is said to be almost periodic in t uniformly in x, $x \in \Lambda$, if $f(t, x(t))$ is continuous in t, x for $t \in I$, $x \in \Lambda$ and if for any $\zeta > 0$, it is possible to find an $l(\zeta) > 0$

such that in an interval of length $l(\zeta)$ there is a τ such that the inequality $|f(t+\tau, x) - f(t, x(t))| \leq \zeta$ is satisfied for all $t \in I$, $x \in \Lambda$.

Definition 7.1.3

The solution $x(t)$ of the equation (7.1) with $f(t,0) = 0$ is said to be stable if for each $\varepsilon > 0$ and $t_o = 0$, there exists $\delta \geq 0$ such that $\|x_0\| < \delta$ and $t \geq t_0$ imply $\|x(t, t_0, x_0)\| < \varepsilon$.

Definition 7.1.4

The equilibrium point of the equation (7.1) is said to be exponentially stable, if there exist two positive constant α and β which are independent of the initial values such that for sufficiently small initial values the inequality

$$\|x(t)\| \leq \beta \exp(-\alpha(t-t_0))\|x(t_0)\|$$

is satisfied for all $\|x(t_0)\| \leq \varepsilon$ and $t \geq t_0$. The largest constant α which may be utilized is called the rate of convergence. If the initial values are sufficiently large, then it is said to be globally exponentially stable.

Definition 7.1.5

The solution $x(t)$ of the equation (7.1) is said to be bounded if given $h > 0$, there exists a constant $k > 0$ such that $\|x(t, t_0, x_0)\| < k$ whenever $\|x_0\| < h$, $t \geq t_0$.

Definition 7.1.6

Consider the system :

$$X' = AX - B\Phi(\sigma) + P(t, X), \quad \sigma = C^{\star}X, \tag{7.2}$$

where A is a stable $n \times n$ real matrix, B and C are constant $n \times m$ real matrices (with $C^{\star}$ as the transpose of C), $\Phi(\sigma) = col\Phi_j(\sigma_j), (j = 1, 2, ..., m)$ and $P(t, X)$ is a continuous n-vector function.

The system (7.2) is said to be dissipative if in the solution space x- space, there is a bounded closed set $\mathcal{F}$ such that (i) $x(t_0) \in \mathcal{F}$ implies $x(t) \in \mathcal{F}$ for all $t \geq t_0$; (ii) any solution $x(t)$ starting at any time enters $\mathcal{F}$ at some time t_0 and (iii) there is at least one solution $x_0(t) \in \mathcal{F}$ which is bounded for all t in $\mathbb{R}$. If $\mathcal{F}$ is a uniformly bounded set, then the system (7.2) is said to be uniformly dissipative.

Stability is one of the central properties in system theory and engineering. From a practical point of view, one of the most important properties that a system must satisfy is that it has to be stable, since the system is otherwise useless and potentially dangerous. The theory of stability has got rich results and could be widely used in concrete problems of the real world. In some cases, a system may be stable or asymptotically stable in theory but it is actually unstable in practice because the stable domain or the domain of attraction is not large enough to allow the desired deviation to cancel out. On the other hand, sometimes the desired state of a system may be mathematically unstable and yet the system may oscillate sufficiently near this state that its performance is acceptable, that is, it is stable in practice (Lakshmikantham *et al.*, (1989), (1991)). Extreme stability, i.e. when the difference of each pair of solutions tends to zero as time infinitely increases (convergence of solutions) is also of practical importance.

The convergence property of systems that are stable is important both theoretically and in applications since, small perturbations from the equilibrium point imply that the trajectory will return to it when time goes to infinity. In some applications, it is still not sufficient to know that the trajectories will converge to the equilibrium point at infinite time; but there is a need to estimate how fast the trajectories approach the equilibrium point. The concept of exponential stability can be used for this purpose. Knowing when a system is exponentially stable provides an explicit bound on the trajectory state at any time. Dissipative systems form an important class of systems observed in reality. Their main feature is the presence of mechanisms of energy reallocation and dissipation. Interaction of these two mechanisms can lead to appearance of complicated limit regimes and structures in the system.

It is a well known fact that delays in control loops may lead to bad performance or instabilities, and the study of time delay systems has attracted large interest. One of the problems one may have to deal with, is the possible uncertainties in the value of these delays. Differential equations with delay have many things in common with corresponding equations without delay. Therefore, many results from the stability theory for systems without delay were extended and adjusted to the corresponding equations with delay, see for instance Hale (1971) and Hale and Lunel (1993).

One of the basic methods for investigation of system stability is the second method of Lyapunov. Its application to systems with delay has been developed in two directions.

a. The first direction implies use of finite dimensional functions with an additional condition for the derivative. This is a so called B.S. Razumikhin condition [see Khusainov (1988)].

b. The second method is a Lyapunov-Krasovskii functional method, which has had more comprehensive theoretical ground [see Khusainov (1998)].

Geometrical meaning of Lyapunov function in this regard involves finding the system of closed surfaces that contain the origin and are converging to it. The vector field of motion equations should be directed inside the areas limited by such surfaces. If a solution gets into such area limited by the surface, then it will never leave it again. These surfaces form level surfaces of a Lyapunov function.

For systems without delay, the speed vector on level surfaces is determined only by the present moment of time, i.e. by the point lying on the given surface. The speed in equations with delay depends on the previous history as well: i.e. it depends on the point $x(t-\tau)$, which is usually hard to find. Therefore it is logical to require negative definiteness of Lyapunov function derivative uniformly by the variable $x(t-\tau)$. However, this leads to an excessively sufficient character of the theorems, which in turn makes them inefficient for applications. Because of this Razumikhin (see for instance Khusainov (1998)) suggested that a previous history $x(t-\tau)$ should lie inside the level surface $V(x,t)=\alpha$ in order to estimate the full derivative along systems solutions.

Lyapunov's Method

The study of qualitative behaviour of solutions of differential equations is not certainly new. Most investigations in this direction are of a local character. The behaviour of solution is studied in a sufficiently small neighbourhood of a given solution, e.g. in a neighbourhood of a stationary point or of a periodic solution. The solution becomes different if the investigation is made in the large. In this case, the examined system and a certain domain are given and one has to study all the solutions which are situated in this domain or to find all solutions of a given family which are situated in this domain.

The first direct reference as far as we know towards this approach is the work of Poincare. Ever since this work appeared, there has been an intensified interest among researchers to explore its richness. There is a substantial amount of literature dealing with numerous qualitative behaviour of solutions of differential equations. These have been summarized in the monographs of Coddington and Levinson (1955), Krasovskii (1963), Hahn (1963) and (1967), Halanay (1966), Reissig *et al.*, (1974), Rouche *et al.*, (1977), Rouche and Mawhin (1980). Lyapunov (1892) proposed a fundamental method for studying the problem of stability by constructing what is today known as Lyapunov functions. By a Lyapunov function is meant a function $V(x,t)$ defined in some region or the whole state phase that contain the unperturbed solution $x = 0$ for all $t > 0$ and which together with its derivative $\dot{V}(x,t)$ satisfy some sign definiteness. It is noteworthy that if V and its derivative are of the same sign, then we have what is known as instability, a subject which is not of interest to us here.

The existence of a Lyapunov function V that satisfies conditions of the Lyapunov's theorem on stability and asymptotic stability has been studied by a number of authors. The first results of a general nature were obtained by Massera ((1949) and (1956)) who proved that if $f(t,x)$ in the system (??) of perturbed motion are periodic in t and continuously differentiable, then there exists a continuously differentiable Lyapunov function $V(t,x)$ in a neighbourhood of the asymptotically stable unperturbed motion $x(t;x_0,t_0) = 0$. For the case in which $f(t,x)$ is continuously differentiable, Malkin (see Krasovskii (1963), pg 18) gave necessary and sufficient conditions for the existence of a continuously differentiable Lyapunov function $V(t,x)$ in some neighbourhood of an asymptotically stable unperturbed trajectory. We can also mention the work of Haddock (1974) in this direction.

Barbashin and Krasovskii (see Reissig *et al.*, (1974)) gave conditions that ensure the existence of a Lyapunov function $V(t, x)$ throughout phase space $-\infty < t < \infty$ for global stability. The papers of Kurzweil (1956) and Massera (1956) contain interesting results as they showed that if $f(t, x)$ is only continuous, there does exist a Lyapunov function V, which is as smooth as desired, if the solution is asymptotically stable. LaSalle (1960) introduced an invariance principle and discussed the asymptotic behaviour of solutions of an autonomous ordinary differential equation by a Lyapunov function for which its derivative is non positive definite. Hale (1965) extended the invariance principle to an autonomous functional differential equation with finite delays. However, his theorem based on Lyapunov functionals does not seem to be practical in some cases. Afterward, Haddock and Terjeki (1983) developed the invariance principle to the autonomous case by the Razumikhin method. However, they pointed out that their results could not be extended to non-autonomous systems without extensive modifications. More exposition in this regard can be found in LaSalle (1968), LaSalle and Lefschetz (1961) and Lefschetz (1965). Other contributions in this area in this direction include Abou-El-Ela and Sadek (1992, 1998), Afuwape and Omeike(2010), Omeike (2010a, 2010b), Sadek (2002, 2003) and Tunç (2010a, 2010b, 2010c, 2011, 2012a, 2012b, 2013a, 2013b).

Frequency Domain Approach

In an effort to overcome difficulties that arose from the practical construction of appropriate Lyapunov function for each equation under consideration, in the problem of absolute stability (see for instance Liao (1993) and Zhang (1989)), Lure and Postnikov (see Boyd *et al.*,(1994)) proposed a method which was the origin of the frequency domain technique. They proposed for the first time that a Lyapunov function of the type "a quadratic form plus the integral involving the nonlinearity" be used in order to determine conditions which ensure stability. Lurie (1951) considered the following system

$$\frac{dx}{dt} = Ax + b\varphi(\sigma)\sigma = c^*x, \tag{7.3}$$

where A is an $n \times n$ constant matrix, b and c are n-vectors, c^* denotes the transpose of c and $\varphi(\sigma)$ is a real continuous function. He posed the question: Find conditions on A, b and c in order for the trivial solutions of the equation (7.3) to be globally asymptotically stable for some constants μ_1 and μ_2 satisfying

$$\mu_1\sigma^2 \leq \sigma\varphi(\sigma) \leq \mu_2\sigma^2, \ \varphi(0) = 0.$$

This problem attracted the attention of many researchers (see for instance Aizerman and Gantmancher (1964)) most of whom used the method of Lure resolving equations. However, we single out the work of Popov (1962), Kalman (1963) and Yacubovich (1964) on which the solution to this problem rests (see also Williams (1970) and Rantzer (1996)). A more comprehensive survey on the frequency domain method can be found in Taylor and Narenda (1970, 1973) and Leonov *et al.*, (1996).

Popov (1962), expressed his sufficient conditions not in terms of resolving equations but in terms of frequency response of the linear part of the system.and gave the following:

Popov (1962)

Suppose that the matrix A in the equation (7.3) *is stable and that the continuous function φ satisfies*

$$\varphi(0) = 0, \ 0 < \sigma\varphi(\sigma) < k\sigma^2 \ \textit{for} \ \sigma \neq 0$$

and for some constant k. Suppose further that there is a non negative number q such that

$$Re\{(1 + i\omega q)[c^*(i\omega I - A)^{-1}b]\} > 0 \tag{7.4}$$

for all real ω, where I denotes the $n \times n$ unit matrix. Then the trivial solution of the equation (7.3) *is globally asymptotically stable.*

The inequality (7.5), known as Popov criterion is quite powerful in the sense that all results connected with the Lyapunov function, consisting of a quadratic form plus an integral of the nonlinear term, are subsumed by it.

Kalman's result (1963) proved the existence of the number q in the Popov criterion and specifically showed that the weakened Popov condition given by

$$Re\{(1 + i\omega q)[c^*(i\omega I - A)^{-1}b]\} \geq 0 \tag{7.5}$$

for all $0 \leq \omega < \infty$, supplemented by the condition for stability, is necessary and sufficient for the existence of a Lyapunov function of the type "a quadratic form plus an integral of the nonlinear term".

Yacubovich (1964), making use of his research on special matrix inequalities, established the converse of Popov (1962). He proved that if the Popov criterion, together with some additional conditions is satisfied, then there exists a Lyapunov function of "a quadratic form type plus an integral of the nonlinear term". He solved the problem of the existence of a positive definite symmetric matrix B posed by Kalman (1963). The result of Yacubovich (1965) has been generalised in numerous directions by many authors, and used in deriving conditions that guarantee existence of some qualitative properties of solutions for some nonlinear differential equations. Of great interest to us is the work of Barbalat and Halanay (1970) in which they obtained necessary and sufficient conditions for Raylieh and Lienard equations to have bounded solutions, which are globally exponentially stable and periodic (or almost periodic), according as the forcing term is periodic (or almost periodic). They first generalised the result of Yacubovich (1965) to the case where matrices B and C are $n \times m$-vectors as:

Barbalat and Halanay (1970)

Consider the system

$$X' = AX - B\varphi(\sigma) + P(t), \quad \sigma = C^{\star}X, \tag{7.6}$$

where A is an $n \times n$ real matrix, B and C are $n \times m$ real matrices with $C^{\star}$ as the transpose of C, $\varphi(\sigma) = col\varphi_j(\sigma_j), (j = 1, 2, ..., m)$ and $P(t)$ is an n-vector.

Suppose that in (7.6), the following assumptions are true:

(i) A is a stable matrix;

(ii) $P(t)$ is bounded for all t in $I\!R$;

(iii) for some constants $\hat{\mu_j} \geq 0, (j = 1, 2, ..., m)$,

$$0 \leq \frac{\varphi_j(\sigma_j) - \varphi_j(\hat{\sigma}_j)}{\sigma_j - \hat{\sigma}_j} \leq \hat{\mu}_j, (\sigma_j \neq \hat{\sigma}_j);$$

(iv) there exists a diagonal matrix $D > 0$, such that the frequency domain inequality

$$\pi(\omega) = MD + ReDG(i\omega) > 0$$

holds for all ω in $I\!R$, where $G(i\omega) = C^{\star}(i\omega I - A)^{-1}B$ is the transfer function and $M = diag(\frac{1}{\mu_j}), (j = 1, 2, ..., m)$. Then, system (7.6) has the following properties;
(I) existence of a bounded solution which is globally exponentially stable;
(II)existence of a solution which is periodic (almost periodic) according as $P(t)$ is periodic (almost periodic).

Dual Systems

The system

$$X' = A_1X - B_1\phi(\bar{\sigma}_1) + P_1(t), \;\; \bar{\sigma}_1 = C_1^{\star}X \tag{7.7}$$

is a dual to the system (7.6), if $A_1 = A^{\star}$, $B_1 = C$, $C_1 = B$ and $P_1 = TP(t)$, where T is a non-singular matrix transformation and $A^{\star}$ is the transpose of A. Let us remark that systems that are dual to each other satisfy the same frequency domain criteria. This assertion was proved by Barbalat and Halanay (1970) whose work opened up a new area in the qualitative study of nonlinear differential equations, as many results on higher order that had been hitherto obtained with the Lyapunov's second method, were improved with the use of the frequency domain technique. We can mention in this direction, the work of Halanay (1972) and Barbalat and Halanay (1974) on the uniform dissipativity of systems which generalised the work of Yacubovich (1964).

Outstanding results that have generalised the result of Barbalat and Halanay (1970) to higher order nonlinear differential equations include, the results of Afuwape (1979, 1984a, 1986) on third order equations; while on fourth order equations, we can mention the work of Afuwape (1985), Adesina (2000) and Afuwape and Adesina (2005). Contributions on uniform dissipativity of solutions include the work of Halanay (1971), Barbalat and Halanay (1971), Barbalat (1973), Barbalat and Halanay (1974) and Afuwape (1978, 1981a, 1984b, 1987) on third order equations. Results on uniform dissipativity of solutions for fourth order equations include Barbalat and Halanay (1971), Halanay (1972), Barbalat and Halanay (1974) and Afuwape (1989, 1991).

bf Kurzweil Theorem on Invariance of Manifold of Flow (Halanay 1969)
Consider
(7.8)
$$X' = AX(t) - BX(t-\tau) + Q\varphi(\sigma(t)) + P(t), \ \ \sigma(t) = C^{\star}X(t),$$

where A is an $n \times n$, B C and Q are $n \times m$ real valued matrices and $P(t)$ is an n-vector. Suppose that the following conditions hold for the system (7.8);
(i) the trivial solution of the system

$$X' = AX(t) - BX(t-\tau) \tag{7.9}$$

is uniformly asymptotically stable;
(ii) $\varphi(\sigma) = Col(\varphi_j(\sigma_j)), (j = 1, 2, ..., m)$ satisfies

$$0 \leq \frac{\varphi_j(\sigma_j) - \varphi_j(\hat{\sigma}_j)}{\sigma_j - \hat{\sigma}_j} \leq \mu_j, \ \ (\sigma_j \neq \hat{\sigma}_j), \ \ \varphi_j(0) = 0,$$

for constants $\mu_j \geq 0, (j = 1, 2, ..., m)$;
(iii) there exist matrices $L \geq 0$, $K > 0$ and a constant $\delta > 0$ such that

$$LK^{-1} + ReLC^{\star}(A + \exp\{i\omega\tau\}B - i\omega I)^{-1}Q \geq \delta I \tag{7.10}$$

for all real ω, where $K = diag(\mu_1, \mu_2, ..., \mu_m)$;
(iv) $P(t)$ satisfies

$$|P(t)| \leq \rho_0$$

for all t. Then the system (7.8) has a bounded solution on $\mathbb{R}$, which is globally exponentially stable. Moreover, if $P(t)$ is periodic (or almost periodic) so also is the bounded solution.
The result of Halanay (1969) is a special case of Kurzweil (1967) and a generalisation of the work of Halanay (1961). Let us remark that if in the system (7.8), the nonlinear term φ involves a delay, i.e. $\varphi(\sigma(t-\tau))$, then the inequality (7.10) changes to
(7.11)
$$LK^{-1} + Re[L\exp\{-i\omega\tau\}C^{\star}(A + \exp\{-i\omega\tau\}B - i\omega I)^{-1}Q] \geq \delta > 0.$$

These results were further used in Halanay (1969), Rasvan (1972a, 1972b, 1973) on the scalar equations of the form

$$\dot{x}(t) + ax(t) + bx(t-\tau) + \varphi(x(t)) = p(t), \tag{7.12}$$

and

$$\dot{x}(t) + ax(t) + bx(t-\tau) + \varphi(x(t-\tau)) = p(t). \tag{7.13}$$

Criteria that made their solutions to be bounded, globally exponentially stable and periodic (or almost periodic) whenever $p(t)$ is periodic (or almost periodic) were obtained. Relatively recently, Afuwape (1999) with the introduction of delay on the nonlinear term, generalised these results to third order nonlinear differential equations of the form:

$$x'''(t) + ax''(t) + bx'(t) + cx(t) + dx(t-\tau) + h(x(t)) = p(t) \tag{7.14}$$

and

$$x'''(t) + ax''(t) + bx'(t) + cx(t) + dx(t-\tau) + h(x(t-\tau)) = p(t). \tag{7.15}$$

Our results in Adesina and Ukpera (2008) are the first on fifth order nonlinear differential equations with delay in which the frequency domain technique was employed. By using the topological degree method, Tejumola and Afuwape (1990) investigate the existence of 2π-periodic solutions of some fifth order nonlinear differential equations with delay, under certain resonant and non resonant conditions. More expository results on the frequency domain approach to delay differential equations can be found in Popov and Halanay (1962), Halanay (1966, 1967), Corduneanu (1973, 1976), Gromova and Pelevina (1978), Haddock and Terjeki (1983), Kolmanovskii and Nosov (1986), Malek-Zavarei and Jamshidi (1988), Kolmanovskii and Myshkis (1992), Kolmanovskii and Shaikhat (1996), Verriest and Fan (1996), Kolmanovskii *et al.*, (1999), Bliman (2000, 2001) and Bartha (2003) used the Lyapunov's approach to study delay differential equations. These cited works contain rich bibliographical information in this field.

8. My Research in Differential Equations

Almost every evolutionary phenomenon encountered in the real world has qualitative behaviour(s) understood in one or another sense. From the theoretical viewpoint, the concept of these behaviours is an underlying principle in numerous practical problems modeled by various kinds of differential equations. Our research activities focus essentially on higher order nonlinear differential equations (with and without delays) with special emphasis on the qualitative behaviour of solutions of these equations. These can be classified into the search for the following properties of solutions of considered nonlinear differential equations:

(a) Boundedness of Solutions;
(b) Exponential Stability of Solutions;
(c) Periodic Solutions;
(d) Almost Periodic Solutions;
(e) Uniformly Dissipative Solutions;
(f) Convergence of Solutions;
(g) Existence of Limiting Regimes;
(h) Stability of Solutions;
(i) Square Integrable Solutions;
(j) Non-Resonant Oscillations.

In this direction, nonlinear equations of third, fourth, fifth and nth orders have been investigated extensively. The study of whichever property varies with the considered equations. The method of investigation also varies. These methods can be divided into:

1. Frequency Domain: This involves the study of position of the characteristic polynomial roots in the complex plane. The equations are reduced to a first order system; transfer functions obtained, which are then used in obtaining matrix inequalities. The method is used in studying properties (a), (b), (c), (d) and (e) above;

2. Lyapunov: This involves constructing a suitable positive definite Lyapunov function whose derivative is negative definite. Lyapunov function in this regard involves finding the system of closed surfaces that contain the origin and are converging to it. The vector field of motion equations should be directed inside the areas limited by such surfaces. If a solution gets into such area limited by the surface, then it will never leave it again. These surfaces form level surfaces of a Lyapunov function. The method is used in studying properties (f), (g) and (h) above;

3. Exponential Dichotomy: The norm of the projection onto the stable subspace of any orbit in the system decays exponentially as t tends to positive infinity and the norm of the projection onto the unstable subspace of any orbit decays exponentially as t tends to negative infinity and furthermore that the stable and unstable subspaces are conjugate. The method is used in studying property (d) above;
4. Topological: This depends on the employment of Leray-Schauder continuation technique to study the spectral properties associated with nonlinear functions of concerned equations. The method is used in studying property (j) above.
5. Square Integrable: Measurable functions that are square-integrable, in the sense of the Lebesgue integral, forms a vector space which is a Hilbert space, provided functions which are equal almost everywhere are identified. The method is used in studying property (i) above.

We have systematically and progressively used the frequency domain method to study uniform dissipative and some qualitative properties of solutions (boundedness, exponential stability, periodicity and almost periodicity) for varieties of third, fourth and fifth-ordered differential equations with various combinations of non-linear terms with and without delays Adesina (2000, 2001, 2003, 2004a, 2004c, 2006, 2012, 2014), Afuwape and Adesina (2000a, 2000b, 2003, 2005a, 2005b), Adesina and Ukpera (2007b, 2008), Afuwape, Adesina and Ebiendele (2006, 2007). In the case of fifth order equations, our attempt using this approach is the first in the literature Afuwape and Adesina (2000a). The analysis involved increases with the order of the equation and the number of the nonlinear function present. The equations that were investigated are important in both theory and practice; because they can be applied to model automatic controls in electronic systems realized by means of R-C filters and also find applications in some three loop electric circuit problems Adesina (2001, 2004a). They are also used in modeling of mathematical problems in economics, epidemiology and biology. Obtained results in this direction have enhanced active research into fifth order nonlinear differential equations, which had hitherto been relatively quiet due to the obvious difficulty in tackling odd ordered differential equations; and also throw more lights on the behaviour of solutions of lower and higher analogous of these equations. The computations and analysis required for the derivation of our results are facilitated by novel advanced techniques.

We have also been able to show for the first time in the literature that convergence results and existence of limiting regime in the sense of Demidovich are provable, for some fifth order nonlinear differential equations with the restoring function and other nonlinear terms not necessarily differentiable Adesina (2007), (Adesina and Ukpera 2007, 2009). Fourth order analogous are treated in Adesina (2012), Adesina and Ogundare (2012). In this direction, the not necessarily differentiable nonlinear functions are only required to satisfy some increment ratios that lie in a closed sub interval of the Routh-Hurwitz interval. The convergence property of systems that are stable is important both theoretically and in applications since; small perturbations from the equilibrium point imply that the trajectory will return to it when time goes to infinity. In some applications, it is still not sufficient to know that the trajectories will converge to the equilibrium point at infinite time; but there is a need to estimate how fast the trajectories approach the equilibrium point. The concept of exponential stability has also been used for this purpose. Knowing when a system is exponentially stable provides an explicit bound on the trajectory state at any time.

Our work on asymptotic stability and boundedness are found in Adesina and Ogundare (2012b) and Ademola, Ogundiran, Arawomo and Adesina (2008), Ademola, Ogundiran, Arawomo and Adesina (2010) and Ademola, Ogundare, Ogundiran and Adesina (2015).

Criteria that generalized and improved existing results on some third and fourth order nonlinear differential equations with different forcing terms have been obtained. In particular, on third order equations, existence of bounded and L^2-solutions for some continuous square integrable functions have been obtained (Ogundare, Ayanjimi and Adesina 2006). We introduced a forcing term and a more general sector condition on the nonlinear term which is differentiable, for a certain Lurie system, and obtained exponential stability, periodicity, almost periodicity and dissipativity results on the solution Adesina (2004a, 2001). By using the method of exponential dichotomy, the Schauder and Horn fixed point theorem, known results on almost periodic solutions to nth order nonlinear differential equations are improved and generalized in Adesina and Ayanjimi (2008) and Adesina (2011a).

We have also paid close attention to the behaviour of stochastic delay differential equations. Of particular interest is the investigation of the effect of stabilization on small departures from deterministic differential equations when the underlying deterministic equation has known asymptotic behaviour (Adesina, Oshinubi, Osilagun and Ajadi 2012). Relatively recently, in Odejide and Adesina (2012), we used the weighted residual method to solve a certain nonlinear fifth order boundary value problems arising in viscoelastic fluid flows.

8.1. Research topics and my contributions

Mr Vice Chancellor, Sir, below are some of our selected contributions to the discipline of mathematics in general and differential equations in particular:

Our exposition (Afuwape and Adesina 2000) on boundedness, exponential stability, periodic and almost periodic solutions via frequency domain technique which is the first work in the literature considers the following fifth order nonlinear differential equations

$$x^{(v)} + ax^{(iv)} + bx^{'''} + cx^{''} + dx^{'} + h(x) = p(t), \tag{8.1}$$

$$x^{(v)} + ax^{(iv)} + bx^{'''} + cx^{''} + g(x^{'}) + ex = p(t), \tag{8.2}$$

$$x^{(v)} + ax^{(iv)} + bx^{'''} + cx^{''} + g_1(x)x^{'} + ex = p(t), \tag{8.3}$$

$$x^{(v)} + ax^{(iv)} + bx^{'''} + f(x^{''}) + dx^{'} + ex = p(t), \tag{8.4}$$

$$x^{(v)} + ax^{(iv)} + \psi(x^{'''}) + cx^{''} + dx^{'} + ex = p(t), \tag{8.5}$$

where the functions that appeared in the equations are real valued and continuous in their respective arguments, and the constants a, b, c, d and e positive. The following are assumed:

(i) the Routh-Hurwitz conditions for stability of solutions of the linear homogeneous equation of the above equations are:

$$a > 0, (ab - c) > 0, (ab - c)c - (ad - e)a > 0, \quad (ab - c)(cd - be) - (ad - e)^2 > 0, e > 0, \tag{8.6}$$

and the consequences of these conditions are:

$$b > 0, c > 0, cd - be > 0, ad - e > 0;$$

(ii) the equations $v^2a - vc + e = 0$ and $v^2 - vb + d = 0$ have two real positive roots given by v_1, v_2 and v_3, v_4 respectively, where

$$v_1 = \frac{1}{2a}[c - (c^2 - 4ae)^{\frac{1}{2}}], \tag{8.7}$$

$$v_2 = \frac{1}{2a}[c + (c^2 - 4ae)^{\frac{1}{2}}], \tag{8.8}$$

$$v_3 = \frac{1}{2}[b - (b^2 - 4d)^{\frac{1}{2}}], \tag{8.9}$$

$$v_4 = \frac{1}{2}[b + (b^2 - 4d)^{\frac{1}{2}}], \tag{8.10}$$

such that $0 < v_1 < v_3 < v_2 < v_4$;
(iii) there exist positive parameters c, d, e, μ_1, μ_2 and μ_3 such that the functions h, g, g_1, f and ψ satisfy respectively the following inequalities

$$e \leq \frac{h(z) - h(\bar{z})}{z - \bar{z}} \leq e + \mu_1, \ (z \neq \bar{z}), \tag{8.11}$$

$$d \leq \frac{g(z) - g(\bar{z})}{z - \bar{z}} \leq d + \mu_2, \ (z \neq \bar{z}), \tag{8.12}$$

$$d \leq \frac{1}{x}\int_0^x g_1(s)ds \leq d + \mu_2, \ x \neq 0, \tag{8.13}$$

$$c \leq \frac{f(z) - f(\bar{z})}{z - \bar{z}} \leq c + \mu_3, \ (z \neq \bar{z}); \tag{8.14}$$

$$b \leq \frac{\psi(u)) - \psi(\bar{u})}{u - \bar{u}} \leq b + \mu_4, \ (u \neq \bar{u}); \tag{8.15}$$

(iv) $\psi = f(0) = g(0) = g_1(0) = h(0) = 0$;
(v) $p(t)$ bounded in $\mathbb{R}$.

Theorem

Suppose that in the equation (8.1), *a, b, c and d are positive constants, with functions h and p continuous in their respective arguments, and satisfying assumptions (iii) equation* (8.11), *(iv) and (v) such that the inequality*

$$\mu_1 \leq M_1 \leq \left(b - \frac{c}{a}\right)\left(\frac{c + (c^2 - 4ae)^{\frac{1}{2}}}{2a}\right) + d - \frac{e}{a}$$

is satisfied, where M_1 is a constant. Then the equation (8.1) *has a unique solution $x(t)$ which is*
(P_α) *bounded in* $I\!R$ *together with its first four derivatives* $x'(t), x''(t)$, $x'''(t)$ *and* $x^{(iv)}(t)$;
(P_β) *globally exponentially stable, together with its first four derivatives ;*
(P_γ) *periodic (or almost periodic) according as $p(t)$ is periodic (or almost periodic).*

Several other results under this heading which we cannot discuss in details include Adesina (2000) which considers the qualitative behavior of solutions of a certain fourth order nonlinear differential equations and Adesina (2001) which represents further work on the above stated theorem. In Adesina (2004b), a special Lurie direct control system with a single differentiable nonlinear term is considered and forcing term is introduced to to make it a more general system. Necessary and sufficient conditions for the existence of a bounded solution which is globally exponentially stable and periodic (or almost periodic) were obtained. Furthermore, Adesina (2004a), Afuwape and Adesina (2005) and Afuwape, Adesina and Ebiendele (2007) improve, extend and generalise existing results in the literature.

We have also contributed significantly to the study of uniformly dissipative solutions of higher order nonlinear differential equations. Investigations in Afuwape and Adesina (2000b) concerns the uniformly dissipative solutions of fifth order nonlinear differential equations (8.1)-8.4 with $p(t)$ replaced with $p(t, x^{'}, x^{''}, x^{'''},$
$x^{(iv)})$. Criteria that guarantee the existence of uniformly dissipative solutions for these equations are given. One of our earliest

results in this direction is given below.

Theorem

Suppose that in equation (8.1), (*i*) $h(0) = 0$*;*
(*ii*) *there exist constants* $e > 0, \mu_1 \geq 0$ *such that* $e \leq \frac{h(z)}{z} \leq e + \mu_1$, $(z \neq 0)$,
and $e \leq h'(z) \leq e + \mu_1$*;*
(*iii*) *the roots* $\omega_1{}^2$, $\omega_2{}^2$ *of equation* $\omega^4 a - c\omega^2 + e = 0$ *are real with* $0 < \omega_1{}^2 < \frac{c}{a} < \omega_3{}^2 < \omega_2{}^2 < \frac{b}{2} < \omega_4{}^2$*;*
(*iv*) *the function* $p(t, x', x'', x''', x^{(iv)})$ *is bounded for all* t*. Then the equation* (8.1) *has uniformly dissipative solutions if*

$$\lim_{|z|\to\infty} \frac{1}{z^2}\left[\int_0^z h(\zeta)d\zeta - \frac{1}{2}zh(z)\right] \leq 0$$

is satisfied.

We have made extensive contributions in this area, indeed our results in Adesina (2003), Afuwape and Adesina (2003), Adesina (2004c, 2006), Adesina and Ukpera (2007) and Adesina (2011b) have remained most distinct in this regard and are reference points in the literature. The Proceedings in Adesina (2011b) is summarised in what follows.

Consider the Lurie direct control system:

$$(8.16) \qquad \begin{cases} \dfrac{dx}{dt} = Ax + bf(\sigma); \\ \sigma = c^T x, \end{cases}$$

where $A \in \mathbb{R}^{n\times n}$, $c, b \in \mathbb{R}^n$, $f(\sigma) \in F$

$$(8.17) \qquad F = \{f : f(0) = 0, 0 \leq \frac{f(\sigma)}{\sigma} \leq k < +\infty \text{ for } \sigma \neq 0\}.$$

If in the system (8.16), there exists a real number $q \geq 0$, such that

$$(8.18) \qquad Re\{(1 + i\omega q)W(i\omega)\} \geq 0 \text{ for all } \omega \geq 0,$$

where $W(i\omega) = -c^T(i\omega I - A)^{-1}b$, then the zero solution of the system (8.16) is absolutely stable. This is the Popov's criterion (1962) obtained by using the frequency domain method.

In an interesting paper Zhang (1989) (see also Liao (1993)) considered a special case of the system (8.16) where

$$A = \begin{pmatrix} -\lambda & 1 & 0 & \dots & 0 \\ 0 & -\lambda & 0 & \dots & 0 \\ 0 & 0 & -\lambda & 0\dots & 0 \\ \vdots & \vdots & 0 & \ddots & \vdots \\ & & \vdots & & \\ 0 & 0 & \dots & \dots & -\lambda \end{pmatrix}, \ \lambda > 0. \tag{8.19}$$

Popov's criterion was used in obtaining

$$c^T b \leq 0, \ c^T A^{-1} b \geq 0 \tag{8.20}$$

as the necessary and sufficient conditions for the absolute stability of the system (8.16). Adesina (2004b) considers a more general system

$$\begin{cases} \dfrac{dx}{dt} = Ax - bf(\sigma) + P(t); \\ \sigma = c^T x, \end{cases} \tag{8.21}$$

where A is of the form (8.19), $P(t)$ bounded, c, b and f are as given in (8.16) and (8.17).

This work obtains an effective criteria for the existence of uniformly dissipative solutions for the Lurie system (8.21) for which $P(t) = P(t, X)$ i.e.

$$\begin{cases} \dfrac{dx}{dt} = Ax - bf(\sigma) + P(t, X); \\ \sigma = c^T x, \end{cases} \tag{8.22}$$

where $P(t, X)$ is bounded, A is of the form (8.19), c, b and f are as given in (8.16) and (8.17). Furthermore, under a real similarity transformation, it is shown that the Lurie system (8.22) and its dual have equivalent frequency domain conditions. Obtained results generalised earlier results of Adesina (2004b) which itself is an improvement on the works of Zhang (1989) and Liao (1993).

The main results of this work is now advertised as:

Theorem

Suppose that in the system (8.22), *the following assumptions are true:*

(i) *A is a stable matrix;*

(ii) *There is a positive constant* $\rho_0 > 0$ *such that for all* t, X

$$|P(t,X)| \le \rho_0$$

(iii) f *is a differentiable real function that satisfies the equation and the following inequality*

$$\lim_{|\lambda|\to\infty} \frac{1}{\lambda^2} \times \frac{\theta}{\mu}\left[\int_0^z f(\zeta)d\zeta - \frac{1}{2}f(z)\right] \le 0.$$

If in addition to the above, $c^T b < 0$*, then the solutions of the system* (8.22) *are uniformly dissipative provided that for some parameters* τ*,* μ*,* λ*,* η *and* θ

$$2\frac{\theta}{\tau} < 3;$$

$$0 < \mu < \frac{\left(3 + \frac{2\theta}{\tau}\right)\lambda^2}{c^T b\lambda - c_1 b_2(1 + 2\lambda)}$$

Theorem

If there exists a real similarity transformation which transforms the matrix $\tilde{A}$ *of the system*

$$\text{(8.23)} \qquad \begin{cases} \dfrac{dx}{dt} = \tilde{A}x - \tilde{b}f(\tilde{\sigma}) + \tilde{P}(t,X); \\ \\ \sigma = \tilde{c}^T x, \end{cases}$$

where $\tilde{A} \in I\!R^{n\times n}$*,* $\tilde{b}, \tilde{c} \in I\!R^n$ *and* $f \in F$ *into the form presented in the Generalized Theorem of Yacubovich, then the frequency domain inequalities for both the systems* (8.22) *and* (8.23) *are equivalent and consequently system* (8.23) *has a uniformly dissipative solutions.*

Mr Vice Chancellor, Sir, our research has also been on the convergence of solutions of differential equations of higher orders. Here, we consider the following differential equation

$$\text{(8.24)} \qquad x' = f(x) + p(t),$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ and $p : \mathbb{R}^1 \to \mathbb{R}^n$, with f and p continuous for $x \in \mathbb{R}^n$ and $t \in \mathbb{R}^1$, respectively. The solutions of the equation (8.24) are said to converge if for every pair of solutions $x(t)$ and $y(t)$ of the equation (8.24),

$$\lim_{t\to\infty}[x(t)-y(t)]=0.$$

A fundamental problem is to seek conditions on f and p which will make all solutions of the equation (8.24) to converge. This problem has been studied by many authors, namely, Levinson (1942), Cartwright and Littlewood (1947), Reuter (1951a), Yoshizawa (1954), Ezeilo (1965b, 1966, 1973), Swick (1969), (1971), Swick (1977), Tejumola (1972) and Afuwape (1981b, 1983, 1988a), Adesina (2007), Adesina and Ukpera (2007), Adesina and Ogundare (2012a). In these works, a Lyapunov function V was constructed to measure the distance between solutions and to show that this distance must approach zero for large enough t. This situation is similar to the Lyapunov theorems concerning asymptotic stability since for solution x and y of the equation (8.24), it can be shown that

$$V(x,y)\geq a(||x-y||)$$

and that $V'_{(8.23)}$, the time derivative of V with respect to the product system

$$\begin{aligned} x' &= f(x)+p(t),\\ y' &= f(y)+p(t), \end{aligned}$$

satisfies $V'_{(8.23)}(t,x,y)\leq -b(||x-y||)$, where $a(r)$ and $b(r)$ are positive definite and both tend to infinity as $r\to\infty$.

The problem which arises with this approach is the same as the problem which arises when trying to find a Lyapunov function to establish asymptotic stability of solutions of the equation (8.24). In other words, it is very often easier to find a function V, for which $V'_{(8.23)}$ is negative definite with respect to closed set Ω not equal to the origin, than it is to find a function V for which $V'_{(8.23)}$ is negative definite with respect to the origin.

Our major contributions in the above stated area which have remained the most general result appeared in Adesina (2007) where two fundamental questions which do not appear to have received any attention in the literature on nonlinear fifth order differential equations, namely, whether convergence of solutions to nonlinear fifth order differential equation can be concluded when the restoring term satisfies certain increment ratio and whether convergence result is provable when the nonlinear functions are not necessarily differentiable are answered. Furthermore, results in Adesina and Ukpera (2007) generalise the above work, while Adesina and Ogundare (2012a) deals with fourth order nonlinear differential equations with four nonlinear terms. d

We have also ma~~k~~e contributions in the study of asymptotic stability, boundedness and ultimate boundedness of solutions through the Lyapunov's Second Method. Ademola, Ogundiran, Arawomo, and Adesina (2008) considers the problem of uniform asymptotic stability for a certain third order nonlinear differential equation. Criteria under which all solutions $x(t)$, its first and second derivatives tend to zero as t tends to infinity are given. Similarly, Ademola, Ogundiran, Arawomo, and Adesina (2010) provides sufficient conditions for the existence of boundedness and ultimate boundedness solutions for a certain third order nonlinear differential equations. A new and complete Lyapunov function is proposed to discuss these properties. Furthermore, Adesina and Ogundare (2012a) deals with the issue of globally asymptotically stable solutions for a certain fourth order nonlinear differential equation. Boundedness and ultimately boundedness of solutions are also considered. The nonlinear functions involved are assumed not necessarily differentiable, but only restricted to satisfy certain increment ratios that lie in the closed sub-interval of the Routh-Hurwitz interval.

Two of our works Adesina and Ukpera (2009) and Adesina (2012) focus on limiting regime in the sense of Demidovich. While the outcome of the former reveals that existence of a limiting regime in the sense of Demidovich is provable for fifth order nonlinear differential equations and also shows that this limiting regime had periodic or almost periodic properties in t uniformly in all the displayed arguments. The later considers a more general fourth order nonlinear differential equation of the form:

$$(8.25) \quad x^{(iv)} + \phi(x''') + f(x'') + g(x') + h(x) = p(t, x, x', x'', x'''),$$

where nonlinear functions ϕ, f, g, h and p are continuous in the arguments displayed explicitly. It must be emphasized here that until our contribution, the equation 8.25 had remained intractable due to (i) the number of the nonlinear terms ϕ, f, g and h simultaneously involved and (ii) the form of the functions ϕ and f (see for instance Tejumola (2006)).

Theorem

Suppose that

(i) *there are positive constants* a, a_0, b *and* b_0 *such that*

$$a \leq \frac{\phi(w_2) - \phi(w_1)}{w_2 - w_1} \leq a_0, \quad w_2 \neq w_1,$$

$$b \leq \frac{f(z_2) - f(z_1)}{z_2 - z_1} \leq b_0, \quad z_2 \neq z_1,$$

(ii) *for any* ζ, η, $(\eta \neq 0)$, *the incrementary ratios for* h *and* g *satisfy*

$$\frac{h(\zeta + \eta) - h(\zeta)}{\eta} \in I_0,$$

$$\frac{g(\zeta + \eta) - g(\zeta)}{\eta} \in I_1;$$

where I_0 *and* I_1 *are closed intervals defined respectively by*

$$I_0 \equiv \Big[\Delta_0, K_0\big[\frac{[(ab - c)c]}{a^2}\big]\Big],$$

$$I_1 \equiv \Big[\Delta_1, K_1\big[\frac{[(a^2 d + c^2)]}{ac}\big]\Big]$$

with a, b, c, d, $\Delta_0 > 0$, $\Delta_1 > 0$, $0 < K_0 < 1$ *and* $0 < K_1 < 1$ *as constants.*

(iii) *there exists a positive constant* B_0 *such that*

$$|Q(t)| \leq B_0, \forall t \in I\!R,$$

where

$$Q(t) = \int_0^t q(\tau) d\tau;$$

(iv) *for a continuous function $\vartheta(t)$, the inequality*

$$|r(t, x_2, y_2, z_2, w_2) - r(t, x_1, y_1, z_1, w_1)| \\ \leq \vartheta(t)(|x_2 - x_1| + |y_2 - y_1| + |z_2 - z_1| + |w_2 - w_1|)$$

for arbitrary $t, x_1, y_1, z_1, w_1, x_2, y_2, z_2, w_2$ holds and satisfies

$$\int_{-\infty}^{\infty} \vartheta^{\beta} dt < \infty,$$

for some constant β in the range $1 \leq \beta \leq 2$.

Then there exists a unique solution $X(t)$ of the equation 8.25 which is a limiting regime in the sense of Demidovic.

Theorem

In addition to hypotheses (i)–(iii) of the above theorem, suppose that there is a solution $X(t)$ of the equation 8.25 satisfying the inequality and that $Q(t)$ is uniformly almost periodic in t for $(x^2 + y^2 + z^2 + w^2)^{\frac{1}{2}} \leq D_1$. Then the solution $X(t)$ is uniformly almost periodic in t. Furthermore, if $Q(t)$ is periodic (respectively almost periodic) with a period ω say, and $r(t, x, y, z, w)$ is periodic (respectively almost periodic) in t with period ω, then the solution $X(t)$ is periodic (respectively almost periodic) in t with period ω.

Furthermore, Afuwape and Adesina (2005) considers the problems of bounds for the mean value of solutions of two nonlinear differential equations of third order. It is shown that there are bounds for the mean values of solutions to these equations as well as their first two derivatives under the condition that the solutions considered are bounded and globally exponentially stable. These bounds make it possible to give interesting conclusions about the existence of some qualitative properties of solutions including periodic solutions whenever the forcing force is periodic and has zero mean value.

Mr Vice Chancellor, Sir, Odejide and Adesina (2012) dwells on the method of weighted residuals to solve nonlinear fifth order boundary value problems arising in viscoelastic fluid flows. A numerical example is presented to illustrate the accuracy, efficiency and reliability of this method. Ajadi, Adesina and Jejeniwa (2013) considers approximate analytical expressions for the solutions of a system of coupled nonlinear reaction-diffusion equations using the homotopy perturbation method(HPM) framework. The effectiveness of this method is elucidated by applying this procedure to the reaction diffusion Brusselator model in one and two space dimensions. For the cases considered, the study reveals that the HPM reduces computational work and converges rapidly to its closed-form solutions. Graphical demonstration of these solutions sheds more lights on the behavior of the system. This is our effort in the area of partial differential equations.

Our efforts on square integrable solutions produced Ogundare Ayanjimi and Adesina (2006) where the work proves the existence of bounded and L^2- solutions to a certain third order nonlinear equations with a continuous square integrable function. Sufficient conditions which guarantee that all solutions of the considered equation are bounded in $L^2[0,)$ are obtained. Adesina and Ayanjimi (2008) and Adesina (2011) are the outcome of this study. The first paper considers a nonlinear almost periodic differential equations and its homogeneous differential equation with $A(t)$ an almost periodic matrix, and the function f almost periodic in t uniformly with respect to x on any compact set. By applying the Schauder fixed point theorem and the method of exponential dichotomy, known results in the literature are generalised and improved. The second paper employs the Horn's fixed point theorem and the method of generalised exponential dichotomy to study the solutions of a general nonlinear almost periodic differential equations and its homogeneous differential equation.

Mr Vice Chancellor, Sir, Adesina and Ukpera (2008), Adesina *et al.*, (2012), Adesina (2014) and Ademola, Ogundare, Ogundiran and Adesina (2015) are important contributions in the area of delay differential equations. As an illustration, Ademola, Ogundare, Ogundiran and Adesina (2015) gives sufficient conditions that guarantee the existence of bounded, stable and periodic solutions for certain third-order nonlinear differential equations with multiple deviating arguments using the Lyapunov's second method by constructing a complete Lyapunov function are given. The work considers

$$\dddot{x}+\sum_{i=1}^{n} f_i(t,x,x(t-\tau_i(t)),\dot{x},\dot{x}(t-\tau_i(t)),\ddot{x},\ddot{x}(t-\tau_i(t)))$$
$$+\sum_{i=1}^{n} g_i(\dot{x}(t-\tau_i(t)))+\sum_{i=1}^{n} h_i(x(t-\tau_i(t)))$$
$$=\sum_{i=1}^{n} p_i(t,x,x(t-\tau_i(t)),\dot{x},\dot{x}(t-\tau_i(t)),\ddot{x},\ddot{x}(t-\tau_i(t)))$$

where f_i, g_i, h_i and p_i are continuous functions in their respective arguments on $\mathbb{R}^+ \times \mathbb{R}^{3n+3}, \mathbb{R}, \mathbb{R}^+$ and $\mathbb{R}^+ \times \mathbb{R}^{3n+3}$ respectively with $\mathbb{R}^+ = [0,\infty)$ and $\mathbb{R} = (-\infty,\infty)$. The dots indicate differentiation with respect to the independent variable t. Our results in this work are new. In our observation, this is the first paper where both the functions f_i and the forcing term p_i will contain sum of multiple deviating arguments. A major result in the work is given below:

Theorem

Further to the assumptions on the functions f_i, g_i, h_i and τ_i, suppose that for all i, $(i = 1,2,3,\cdots,n)$ $a_i, \delta_i, c_i, B_i, \rho$ and γ are positive constants and for all $t \geq 0$:

(i) $\frac{f_i(\cdot)}{z} \geq a_i$ *for all* $z \neq 0$;
(ii) $b_i \leq \frac{g_i(y)}{y} \leq B_i$ *for all* $y \neq 0$;
(iii) $h_i(0) = 0$, $\frac{h_i(x)}{x} \geq \delta_i$ *for all* $x \neq 0$;
(iv) $h_i'(x) \leq c_i$ *for all* x *and* $a_ib_i - c_i > 0$;
(v) $\tau_i(t) \leq \gamma$, $\tau_i'(t) \leq \rho$, $\rho \in (0,1)$; *and if*

$$\gamma < \min\left\{\sum_{i=1}^{n}\delta_i(B_i+c_i)^{-1},\ \sum_{i=1}^{n}(a_ib_i-c_i)A_3^{-1},\ \sum_{i=1}^{n}(a_i-\alpha)A_4^{-1}\right\}$$

where

$$A_3 := (B_i+c_i)(\alpha+a_i)+c_i(1-\rho)^{-1}(\alpha+\beta+a_i+2)$$

and

$$A_4 := 2[2(B_i+c_i)+B_i(1-\rho)^{-1}(\alpha+\beta+a_i+2)$$

then the trivial solution of the equation is uniformly asymptotically stable.

Finally, we also made contributions on non resonant oscillations (Ukpera and Adesina 2010, 2012) where it is noted vector results for boundary value problems of fourth order nonlinear differential systems appear not to have received much, if any, attention, since the theory for the scalar forms are currently still being developed themselves. Notwithstanding, it is important both from the mathematical and practical point of view, to attempt to provide vector versions to available results in the scalar case which appear on the horizon, which is by no means a trivial transition. This is our prime motivation for this work. By making use of the spectral properties associated with the nonlinear terms of the differential equations considered, this work extends to fourth order nonlinear systems of differential equations some results given in the literature for systems of third order equations. Vector versions of some existence results are presented using some hypothesis analogous to the Routh-Hurwitz conditions. Obtained results in this work are novel.

9. Future Research Plans

Aside from continuing working in the areas where I have already published, my immediate future research plans include studying qualitative behaviour of solutions of the following:
(i) functional differential equations with various delay types;
(ii) iterative differential equations;
(iii) impulsive differential equations;
(iv) trinomial delay differential equations;
(v) delay difference and integrodifferential equations.

10. Conclusion

Mr Vice Chancellor, Sir, Distinguished Ladies and Gentlemen, one of the most protracted and controversial debate in Nigeria is the way government handles the problems of education in the country. No nation can make any meaningful progress if the educational sector is neglected; and there would be no technological advancement if the growth indices of mathematical sciences are in constant threats. It is therefore not too difficult to see the irrevocable relationship and understanding between Mathematics and development in a nation that earnestly seeks a deserving and rightful place in the committee of nations. Mathematics is a language and science of patterns. As a language of patterns, Mathematics is a means for describing the world in which we live. In its symbols and vocabulary, the language of Mathematics is a universal means of communication about relationships and patterns. As a science of patterns, Mathematics is a mode of inquiry that reveals fundamental understandings about order in our world. This mode of inquiry relies on logic and employs observation, simulation, and experimentation as means of challenging and extending our current understanding. Thus Mathematics continues to grow at a rapid rate, spreading into new fields and creating new applications in its open-ended search for patterns. Several factors-growth of technology, increased applications, impact of computers, and expansion of Mathematics itself have combined in the past century to extend greatly both the scope and the application of the Mathematical Sciences. There are needs for changes to be effected and reflected in our universities if our students are to be well prepared for tomorrow's world.

For Nigeria to be on the path of advancement, every child must have the right to be taught by a highly qualified teacher of Mathematics, one who is knowledgeable in content, who understands how students learn and uses appropriate instructional methods. Every child must have the opportunity for the Mathematics required for an economically secure future and no single test should limit future opportunities to learn Mathematics. I am of the opinion that Mathematics education in a country like ours should be strengthened. Mathematicians and Mathematics educators should strengthen their relationship with a view to exchanging scientific and pedagogical 'handshake' for the development of Mathematics in Nigeria.

To be well informed as adults and to have access to desirable jobs, graduates today require an education in Mathematics that goes far beyond what was needed by graduates in the past. All students must therefore develop and sharpen their skills; deepen their understanding of mathematical concepts and processes; and hone their problem-solving, reasoning, and communication abilities while using Mathematics to make sense of and to solve compelling problems. All stakeholders in this regard should work together to understand success and for whom, and when we are less successful, with whom. These ends require governmental resources to charter a new way to Mathematics relevant in our country. We must teach the private and public sectors that research in mathematics is not equivalent to other researches; and also attack incorrect public perceptions about Mathematics and its teachers in the media. It is time that we accept no bias against those who teach, study and learn Mathematics.

Government should come up with a policy to give special priority to research in Mathematics, especially, those research that are targeted at solving problems facing us as a Nation.

Federal and State governments, and the private sector should come to the aid of the National Mathematical Centre, Abuja. Centres like this elsewhere are well funded and thus contribute in no small measure to technological advancement and nation building. Government should establish a unit made up of university teachers under the presidency to tackle the myriads of problems facing our nation. Given the enormous potential for controversy surrounding decisions related to appointments in key sectors, I will suggest that the Ministry of Education be re-organised and the staff retrained to be non political and non-sectarian. These retrained staff should be insulated from political pressures so they can more easily make difficult decisions without immediate repercussions. To do this, there should be a non-political board of directors appointed for a few years; another would be a system that allows consultants who do not have vested political interests to make decisions under a type of trusteeship for the education sector. Also there should be a "code of honor" among all the major players so that education-related policy is conducted in a way that advances only the interests of the country and the people of Nigeria.

On a final note, apart from my research activities, as a teacher, I have been able to advance and diffuse knowledge through teaching of several Mathematics courses for over eighteen years at the university level. During this period, I have demonstrated a high level of competence in all professional aspects of teaching, such as construction of courses, classroom presentation, tutorials, assignments and grading; innovation in the classroom; commitment to teaching; evidence of intensive and sustained attention to the teaching and learning process; instilling in students the desire to be lifelong learners, and availability to students. I have also been able to undertake in a responsible manner, the academic and administrative tasks assigned to me. I have graduated students at postgraduate level. I serve actively as a reviewer to over twenty reputable international journals (Elsevier, Springer Verlag, Hindawii etc). I also serve as the External Examiner to Department of Mathematics, University of Ibadan.

Mr Vice Chancellor, Sir, I was recently appointed as an Associate Editor to the Journal of Nigerian Mathematical Society-an Elsevier published journal.

11. Acknowledgements

Mr Vice Chancellor, Sir, distinguished audience, this Inaugural Lecture will never be complete without acknowledging those who have contributed in one way or the other to my career advancement.

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Late Professor C.O. Imoru was an outstanding example for me and many generations of Lecturers and Students, a great professor of an impeccable academical status, a person full of nobility and sensibility. His remarks were always made in a kind and warm tone, without meanness, directed to the sensibility of the person who had made a mistake, so that he or she would never be able to repeat it. It is, of course, very hard for me to make, in few words, a portrait of such a personality.

I must say that I lack adequate words to acknowledge the motivation, inspiration and encouragement I received from Dr.(Mrs) L.M. Durosinmi, Dr. Ayo Omitogun, Professors Tola Badejo, G.O. Babalola, S.R.A. Adewusi, V.O. Olarewaju, W.A. Muse, F.O.I. Asubiojo and Mike Adeyeye. In the same vein, I extend thanks to my colleagues and staff members in the Department of Mathematics and also in the Faculty of Science. Professor Titi Obilade has remained a friend, mentor and among others a father. Professor Samuel S. Okoya, another mentor, has been a pillar of support spiritually and academically. Dr. Taiwo Olaiya, Wole Omole and Lekan Ijiyode are also given my gratitude and appreciation for the enormous support I enjoyed from them. They have been wonderful to me. Dr. Suraju Ajadi, a magnificient personality, has been a good brother, a close confidant and a reliable ally.

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I cannot thank enough many students that have passed through me in the class and when I was the Vice Dean, Students' Affairs. It has been very gratifying to associate with you all. Many of my students may claim that they have learned a lot from me but I must say that I have learned so much from them that it would be impossible to adequately express my appreciation to them. I know that the evening breeze will keep caressing the trees tenderly and the trembling trees will also in return keep embracing the breeze tenderly so that we will continue to hear ourselves into the realisation of expanding mutual respect.

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Mr Vice Chancellor, Sir, Distinguished Ladies and Gentlemen, I thank you for your patience and attention.

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