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HOLD EFFECT IN FINITE TORSION OF A COMPRESSIBLE ELASTIC TUBE

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Abstract. We consider the application of complex variable method to elastic problem and investigate the nonlinear effect of finite torsion of a compressible elastic composite layer. We obtain that as a result of finite deformation approach, a tube subjected to torsion decreases in radius giving rise to a "hold effect".

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1. Introduction

Most nonlinear effects in solid mechanics are observed when specimens work in shears. For an instance, materials fore and foremost get into plastic regime under shear. Torsion is one such regime when materials work in shear. (i.e., shear regime is attained when a specimen is subjected to torsion).

It is not accidental therefore that most experiments carried out to establish (or track down) nonlinear phenomena in tubes and rods are carried out under torsion. Such is the case with most pioneering experiments of Wertheim G. about 1857 and Bauschinger J. about 1881 and the theoretical works of Cauchy and Saint Venants betweeen 1853 and 1856 [1].

The application of complex variable method to problems enjoys a sustained interest in continuum mechanics, including in plane problems of elasticity [2].

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This is particularly so, when analytical solutions are desirable. The theoretical formulation of equilibrium problems of finite elasticity, involving a layered medium, is well exposed in an earlier work [3]; it will be highlighted here for self containment.

In this work, we examine the equilibrium problem of a tube, made up of concentric cylinders of different elastic materials, when subjected to torsion. We make use of the theory of complex variable method, developed for the case of finite elasticity [3, 4, 5], and investigate the nonlinear effect of finite deformation of the compressible elastic composite tube.

Problem Setting

Statement of problem: Let Ω be a layer of concentric cylinders, each of which is compressible and of different elastic properties, in a three dimensional euclidean space $\mathcal{E}^3$. i.e., $\Omega \subset \mathcal{E}^3$ and $\Omega = \cup\Omega_m,\ m = 1, 2, 3 \ldots, n,$ where n is any natural number.

We assume ideal contact between these cylindrical layers and consider the finite deformation of the whole layer, from the initial configuration Ω_o into a current configuration, denoted by Ω. The position vector of particles in the initial and current configurations respectively is given as

$$\mathbf{X} = X^1\mathbf{e}_1 + X^2\mathbf{e}_2 + X^3\mathbf{e}_3 = R\mathbf{e}_R + Z\mathbf{k} \tag{1.1}$$

and

$$\mathbf{x} = x^1\mathbf{e}_1 + x^2\mathbf{e}_2 + x^3\mathbf{e}_3 = r(R)\mathbf{e}_r + z\mathbf{k}\,, \tag{1.2}$$

where X^i and x^i, $i = 1, 2, 3$ are the rectilinear coordinates with common base vectors $\mathbf{e}_i$ in Ω_o and Ω respectively; $\mathbf{e}_R, \mathbf{k}$ and $\mathbf{e}_r, \mathbf{k}$ are base vectors in the corresponding material cylindrical coordinates R, Φ, Z and r, ψ, z.

The deformation in consideration is given as

$$r = r(R),\ \psi(R) = \varphi + \theta(R),\ z = Z \tag{1.3}$$

with the boundary conditions

$$r(R_1) = R_1,\ r(R_2) = R_2;\ \theta(R_1) = \alpha,\ \theta(R_2) = 0. \tag{1.4}$$

That is, we are interested in the torsion of a cylindrical layer having a cross-section $R_1 \le R \le R_2,\ \ 0 \le \Phi \le 2\pi$ in the reference configuration. What is its cross section in the current configuration?

2. General geometry of deformation

We observe that for a sufficiently long cylindrical layer, this transformation is essentially a plane deformation of Ω_o into Ω [5,6].

Let Ω be a transversely isotropic medium in three dimensional euclidean space $\mathcal{E}^3$. We look at the equilibrium state of Ω in plane finite deformation.

The deformation of Ω is given by specifying [7] the position vector $\mathbf{X}$ of a particle prior to deformation in the initial (or reference) configuration Ω_o with the boundary Σ_o and its orientation outward normal unit vector $\mathbf{N}$ and the position vector $\mathbf{x}$ in the current configuration Ω with the boundary Σ and its orientation normal vector $\mathbf{n}$:

$$\mathbf{X} = X^1\mathbf{e}_1 + X^2\mathbf{e}_2 + X^3\mathbf{e}_3 \tag{2.1}$$

$$\mathbf{x} = x^1\mathbf{e}_1 + x^2\mathbf{e}_2 + x^3\mathbf{e}_3 \tag{2.2}$$

such that for plane deformation:

$$x^\alpha = x^\alpha(X^1, X^2); \quad x^3 = kX^3; \quad \alpha = 1, 2\,, \tag{2.3}$$

where X^m, x^m are the material coordinates in Ω_o and Ω respectively; $\mathbf{e}_m$ is the orthonormal basis; $m = 1, 2, 3$; k is any real constant.

Let the geometry of deformations be the tensor-gradient of the position vector $\mathbf{x}$ in $\Omega(\mathbf{x})$ taking in the initial configuration $\Omega_o(\mathbf{X})$ [7]. That is, applying the operator of gradient-vector in the reference configuration, $\overset{o}{\nabla} \equiv e_i \frac{\partial}{\partial X^i}$, on the position vector $\mathbf{x}$ in the current configuration, we obtain the tensor-gradient (or *deformation gradient*), $\overset{o}{\nabla}\mathbf{x} \equiv \mathbf{F}$:

$$\mathbf{F} = \mathbf{e}_\alpha \mathbf{e}_\beta \frac{\partial x^\beta}{\partial X^\alpha} + k\mathbf{e}_3\mathbf{e}_3. \tag{2.4}$$

We also consider the *deformative rotation tensor* of the medium:

$$\mathbf{R} = \mathbf{V}^{-1}\mathbf{F} = \mathbf{I}\cos\chi + (1 - \cos\chi)\mathbf{e}_3 \otimes \mathbf{e}_3 - \mathbf{e}_3 \times \mathbf{I}\sin\chi\,, \tag{2.5}$$

where $\mathbf{V}$, such that $\mathbf{V}^2 = \mathbf{F}\mathbf{F}^T$, is the symmetric *left stretch tensor*, arising from the polar decomposition of the deformation gradient, $\mathbf{F} = \mathbf{V}\mathbf{R}$; $\mathbf{I} = \delta_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$ is the unit tensor in $\mathcal{E}^3$ and

$$\cos\chi = \frac{1}{q}\left(\frac{\partial x^1}{\partial X^1} + \frac{\partial x^2}{\partial X^2}\right), \quad \sin\chi = \frac{1}{q}\left(\frac{\partial x^2}{\partial X^1} - \frac{\partial x^1}{\partial X^2}\right),$$

$$q = \sqrt{\left(\frac{\partial x^1}{\partial X^1} + \frac{\partial x^2}{\partial X^2}\right)^2 + \left(\frac{\partial x^2}{\partial X^1} - \frac{\partial x^1}{\partial X^2}\right)^2}. \tag{2.6}$$

For any vector functions ϕ and ψ, here and elsewhere, we denote their dot product, tensor product and cross product respectively by $\phi\psi$ (or $\phi\cdot\psi$), $\phi\otimes\psi$ and $\phi\times\psi$. Also, for the tensor functions Φ and Ψ we denote their dot product, cross product and double dot product respectively as $\Phi\Psi$, $\Phi\times\Psi$ and $\Phi\cdot\cdot\Psi = tr(\Phi\Psi)$.

Static equation for transversely isotropic material

We look at the equilibrium state of Ω in plane finite deformation. For this, we first recall the energy function for an isotropic semi-linear material in finite deformation, proposed by F. John 1960 [3, 5]:

$$W = \mu S_2 + 1/2\lambda S_1^2, \tag{2.7}$$

where S_1 and S_2 are the invariants of the deformation geometry,

$$S_1 = \mathbf{I}\cdot\cdot(\mathbf{V}-\mathbf{I}) = tr(\mathbf{V}-\mathbf{I}) \equiv I_1(\mathbf{V}-\mathbf{I}), \quad S_2 = I_1(\mathbf{V}-\mathbf{I})^2.$$

λ and μ are the Lame constants.

On the basis of (2.7) an energy function has been constructed for a transversely isotropic semi-linear material in the case of plane deformation [3]:

$$W = \lambda_2 S_2 + 1/2\lambda_1 S_1^2 + \lambda_0 S_0 \tag{2.8}$$

where, $S_0 = \mathbf{c}\cdot\mathbf{V}^2\cdot\mathbf{c}$ is an additional invariant of deformation, due to anisotropy. $\mathbf{c}$ is the unit vector characterising the direction of anisotropy. $\lambda_0, \lambda_1, \lambda_2$ are the material constants. In the case of randomly unidirectional fibre reinforced composite or a lamina composite the material constants are the effective moduli [3]:

$$\lambda_2 = \langle\mu\rangle, \quad \lambda_1 = \langle\lambda\rangle + \frac{\langle\frac{\lambda}{(\lambda+2\mu)}\rangle^2}{\langle\frac{1}{(\lambda+2\mu)}\rangle} - \Big\langle\frac{\lambda^2}{(\lambda+2\mu)}\Big\rangle, \quad \lambda_3 = \frac{1}{\langle\frac{1}{\mu}\rangle}, \quad \lambda_o = \lambda_o(\lambda_2, \lambda_3) \tag{2.9}$$

and we note that in the case of degeneracy into isotropy, the energy function (2.8) automatically reduces to the energy (2.7) and accordingly for the effective moduli $\lambda_3, \lambda_2, \lambda_1$, while $\lambda_o = 2(\lambda_3 - \lambda_2)$ vanishes, i.e.

$$\lambda_3 = \lambda_2 = \mu, \quad \lambda_1 = \lambda, \quad \lambda_o = 0. \tag{2.9$'$}$$

For any finite function $\varphi(\vec{\xi}, t) \in \Omega \times [0, T)$, $\langle\varphi\rangle$ denotes its geometric average over Ω with the volume $|\Omega| : \langle\varphi\rangle = \frac{1}{|\Omega|}\int_\Omega \varphi\, d\,\Omega$.

Now, invoking the hypothesis of hyperelasticity of Cauchy, we take the Frechet derivative [5,8] of the energy with respect to the geometry of deformation (the deformation gradient) $\mathbf{F}$ and obtain the *first Piola-Kirchoff* stress tensor $\mathbf{T}_R$, to which it is energy conjugate:

$$\mathbf{T}_R \equiv \frac{\partial W}{\partial \mathbf{F}} = 2\lambda_2\mathbf{F} + (\lambda_1 S_1 - 2\lambda_2)\mathbf{R} + 2\lambda_0\mathbf{cc}\cdot\mathbf{F}\,. \tag{2.10}$$

In the absence of body force, we obtain the static equation and the accompanying boundary condition:

$$\begin{aligned} \overset{o}{\nabla}\cdot\mathbf{T}_R &= 0 \\ \mathbf{f}d\Sigma &= \mathbf{N}\cdot\mathbf{T}_R d\Sigma_o, \end{aligned}$$

where $d\Sigma$ is the element of the boundary in the current configuration on which the force $\mathbf{f}$ acts while $d\Sigma_o$ is the element of the boundary in the reference configuration, with the normal vector $\mathbf{N} = (N_1, N_2)$.

The component form of these relations are:

$$\frac{\partial T_R^{11}}{\partial X^1} + \frac{\partial T_R^{21}}{\partial X^2} = 0; \quad \frac{\partial T_R^{12}}{\partial X^1} + \frac{\partial T_R^{22}}{\partial X^3} = 0; \quad \frac{\partial T_R^{33}}{\partial X^3} = 0 \tag{2.11}$$

and

$$f_1 \frac{d\Sigma}{d\Sigma_o} = N_1 T_R^{11} + N_2 T_R^{21}; \quad f_2 \frac{d\Sigma}{d\Sigma_o} = N_1 T_R^{12} + N_2 T_R^{22}. \tag{2.11$'$}$$

3. Complex variable formulation

Complex variable representation of static equation

We now look at Ω as a subspace of the complex plane $\mathcal{C}$, such that henceforth $\Omega \longrightarrow \Sigma, \Omega_o \longrightarrow \Sigma_o; \Sigma \longrightarrow s, \Sigma_o \longrightarrow S$.

In place of the material coordinates X^1, X^2 and x^1, x^2 we introduce the complex variables:

$$Z = X^1 + iX^2 \in \Sigma_o \;\text{ and }\; z = x^1 + ix^2 \in \Omega \tag{3.1}$$

where, $i = \sqrt{-1}$ is the imaginary unit. Then

$$\frac{\partial}{\partial Z} = \frac{1}{2}\left(\frac{\partial}{\partial X^1} - i\frac{\partial}{\partial X^2}\right); \quad \frac{\partial}{\partial \overline{Z}} = \frac{1}{2}\left(\frac{\partial}{\partial X^1} + i\frac{\partial}{\partial X^2}\right) \tag{3.2}$$

and

$$2\frac{\partial z}{\partial Z} = \left(\frac{\partial x^1}{\partial X^1} + \frac{\partial x^2}{\partial X^2}\right) + i\left(\frac{\partial x^2}{\partial X^1} - \frac{\partial x^1}{\partial X^2}\right)$$

or, in view of (2.6)

$$\frac{\partial z}{\partial Z} = \frac{1}{2} q \exp(i\chi) \quad \text{and} \quad \exp(i\chi) = \frac{\partial z}{\partial Z}\Big/\left|\frac{\partial z}{\partial Z}\right|. \tag{3.3}$$

The Piola-Kirchoff stress tensor (2.10), the static equations (2.11) and the boundary conditions (2.11)$'$ respectively become:

$$T_R^{11} + iT_R^{12} = \phi(q)\exp(i\chi) + 2i\lambda_2\frac{\partial z}{\partial X^2} + 2\lambda_o\left(q\exp(i\chi) + i\frac{\partial z}{\partial X^2}\right),$$

$$T_R^{22} - iT_R^{21} = \phi(q)\exp(i\chi) - 2\lambda_2\frac{\partial z}{\partial X^1}, \tag{3.4}$$

$$P^{33} = P^{33}(a^1, X^2);$$

$$\frac{\partial \Phi(Z)}{\partial \overline{Z}} = -2\lambda_0\frac{\partial^2 z}{\partial X^1 \partial X^1} \tag{3.5}$$

and

$$if\frac{ds}{dS} - \frac{dZ}{dS}\Phi(Z) + 4\lambda_2\frac{dz}{dS} = 2\lambda_0\frac{\partial z}{\partial X^1}in_1, \tag{3.5$'$}$$

where $N = N_1 + iN_2$; ds and dS are the arc elements in the current and reference configurations respectively, and

$$\Phi(Z,\bar{Z}) \equiv \phi(q)\exp(i\chi); \quad \phi(q) = (\lambda_1 + 2\lambda_2)(q-2) + 2\lambda_2 + \lambda_1(k-1)\,. \tag{3.6}$$

$$\phi(q) = (\lambda_1 + 2\lambda_2)\left[q - \frac{1-\nu_o(k-1)}{1-\nu_o}\right]; \quad \nu_o \equiv \frac{\lambda_1}{2(\lambda_1+\lambda_2)}\,. \tag{3.6$'$}$$

Anisotropic expansion of state variables

We note that if Ω were to be an isotropic body, then the right-hand side in (3.5) and (3.5)$'$ would vanish. This implies that λ_0 is a true parameter of anisotropy. So, we dimensionalize it and expand the state variables in this,

$$\beta = \frac{\lambda_0}{\lambda_1 + 2\lambda_2} < 1, \tag{3.7}$$

$$\begin{gathered} z = z_0 + \beta z_1 + \beta^2 z_2 + \beta^3 z_3 + \cdots \\ \Phi = \Phi_0(Z,\bar{Z}) + \beta\Phi_1(Z,\bar{Z}) + \beta^2\Phi_2(Z,\bar{Z}) + \beta^3\Phi_3(Z,\bar{Z}) + \cdots\ , \\ \mathbf{f} = \mathbf{f_0} + \beta\mathbf{f_1} + \beta^2\mathbf{f_2} + \beta^3\mathbf{f_3} + \cdots\ . \end{gathered} \tag{3.8}$$

Putting (3.8) in (3.5) and (3.5)$'$ we obtain the recurrence system for the equilibrium equations and the boundary conditions:

$$\sum_{m=0}^{\infty} \beta^m \mathcal{F}_m = 0, \tag{3.9}$$

$$\sum_{m=0}^{\infty} \beta^m \mathcal{P}_m = 0\,, \tag{3.9$'$}$$

where

$$\mathcal{F}_m \equiv \frac{\partial\Phi_m}{\partial\overline{Z}} + 2(\lambda_1+2\lambda_2)\frac{\partial^2 z_{m-1}}{\partial X^1\partial X^1}; \quad z_k = 0 \text{ if } k<0,\ k = m-1,\ m = 0,1,2,...,$$

$$\mathcal{P}_m \equiv if_m\frac{ds}{dS} - \frac{dZ}{dS}\Phi_m(Z) + 4\lambda_2\frac{dz_m}{dS} - 2\lambda_0\frac{\partial z_{m-1}}{\partial X^1}iN_1\,.$$

Now, if we set every coefficient of the powers of β to zero in equations (3.9) and (3.9)$'$, we shall obtain a recurrence system of equations, i.e., $\mathcal{F}_m = 0$ and $\mathcal{P}_m = 0$; $m = 0,1,2,\ldots$. The first equation in the recurrence system due to (3.9) is Laplacian (i.e., homogeneous) while each of the subsequent ones are Poissonian (i.e., non-homogeneous), with the right hand depending recursively on the solution of the previous equation. Thus, this much is the effect of anisotropy on the medium: and has in no way influenced the fact or exposed the issue of finite deformation. The effect of finite deformation is exposed in what follows.

Boundary value problem at the first level

Now, the bounary value problem at the first level can be written out explicitly. Really, following from expansion (3.8), let

$$z_0 \equiv w = w_1 + iw_2; \quad \Phi_0 \equiv F(Z, \overline{Z}) \quad \text{and} \quad f_0 \equiv h = h_1 + ih_2\,. \tag{3.10}$$

Then the first equation in (3.9) is:

$$\frac{\partial F(Z, \overline{Z})}{\partial \overline{Z}} = 0 \tag{3.11}$$

with the corresponding boundary condition from (3.9)'

$$ikh\frac{ds}{dS} = \frac{dZ}{dS}F(Z, \overline{Z}) - 4\lambda_2\frac{dw}{dS}\,, \tag{3.11$'$}$$

where $h \equiv f_0$ is the specified force per unit length of the current boundary contour.

We recall that by the Cauchy-Riemann equations, relation (3.11) implies that F is an analytic function of only the variable Z, in the finite plane and can then be written as

$$F = F(Z)$$

and by (3.3), (3.6)$'$, (3.8) and (3.10) we have

$$F(Z) = \phi_0(q_0)\exp(i\chi_0); \quad q_0 = 2\left|\frac{\partial w}{\partial Z}\right|; \quad \exp(i\chi_0) = \frac{\partial w}{\partial Z}\Big/\left|\frac{\partial w}{\partial Z}\right|. \tag{3.12}$$

Now, on every contour of the material, the boundary force can be decomposed into its normal h_n and tangential h_s components. Noting that

$$\begin{aligned} h_n &= \mathbf{n}\cdot\mathbf{h} = 1/2(n\bar{h} + \bar{n}h); \\ h_s &= \mathbf{s}\cdot\mathbf{h} = 1/2(s\bar{h} + \bar{s}h); \\ n &= -i\frac{dw}{ds}, \quad s = \frac{dw}{ds}, \end{aligned}$$

we have

$$h_n = \frac{1}{2k}\left[\frac{dZ}{ds}\frac{d\bar{w}}{ds}F(Z) + \frac{d\bar{Z}}{ds}\frac{dw}{ds}\overline{F}(Z) - 2\lambda_2\right], \tag{3.13}$$

$$h_s = \frac{i}{2k}\left[\frac{d\bar{Z}}{ds}\frac{dw}{ds}\bar{F}(Z) - \frac{dZ}{ds}\frac{d\bar{w}}{ds}F(Z)\right]. \tag{3.14}$$

4. Solution of boundary value problem at the first level

Following from the previous sections, in complex variable formulation the deformation under consideration is given by the tansformation

$$X^1 + iX^2 = Z \longrightarrow w = w_1 + iw_2 :$$
$$z_0 = R_0(r)e^{i(\varphi+\theta(r))} = R_0(r)e^{i\theta(r)}\sqrt{\frac{Z}{\bar{Z}}}. \tag{4.1}$$

Let

$$z_0 \equiv w, \ \ R_0(r) \equiv v(r); \ \ Z = re^{i\varphi}\bar{Z} = re^{-i\varphi}.$$

Then

$$r = \sqrt{Z\bar{Z}},\ e^{i\varphi} = \sqrt{\frac{Z}{\bar{Z}}}$$

and

$$\frac{\partial r}{\partial Z} = \frac{1}{2}(Z\bar{Z})^{-\frac{1}{2}}\bar{Z} = \frac{1}{2}e^{-i\varphi}, \tag{4.2}$$

$$\frac{\partial}{\partial Z}e^{i\varphi} = \frac{1}{2}\sqrt{\frac{\bar{Z}}{Z}}\frac{1}{\bar{Z}} = \frac{1}{2r}. \tag{4.3}$$

By (4.1) - (4.3) we have

$$\begin{aligned}
2\frac{\partial w}{\partial Z} &= 2\frac{\partial w}{\partial r}\frac{\partial r}{\partial Z} + 2\frac{\partial w}{\partial e^{i\varphi}}\frac{\partial e^{i\varphi}}{\partial Z} \\
&= \frac{\partial w}{\partial r}e^{i\varphi} + \frac{\partial w}{\partial e^{i\varphi}}\frac{1}{r} \\
&= \left[v'(r) + iv(r)\theta'(r) + \frac{v(r)}{r}\right]e^{i\theta(r)} \\
&= pe^{i\gamma(r)}e^{i\theta},
\end{aligned} \tag{4.4}$$

where

$$pe^{i\gamma(r)} \equiv \left[v'(r) + iv(r)\theta'(r) + \frac{v(r)}{r}\right] = M(Z,\bar{Z}),$$

$$p^2 = |M(Z,\bar{Z})|^2 = M\bar{M} = \left[v'(r) + \frac{v(r)}{r}\right]^2 + [v(r)\theta'(r)]^2, \tag{4.5}$$

$$\cos\gamma(r) = \frac{1}{p}\left[v'(r) + \frac{v(r)}{r}\right], \ \ \sin\gamma(r) = \frac{1}{p}[v(r)\theta'(r)].$$

Compared with (3.3) or (3.12) and $(3.6)'$, when $k = 1$, we have

$$\chi_0 = \theta + \gamma,\ \phi(p) = (\lambda_1 + 2\lambda_2)\left[p - \frac{2(\lambda_1+\lambda_2)}{\lambda_1+2\lambda_2}\right].$$

Then the harmonic function in (3.12) takes the form

$$F(Z) = (\lambda_1 + 2\lambda_2)\left[p - \frac{2(\lambda_1+\lambda_2)}{\lambda_1+2\lambda_2}\right]e^{i(\gamma+\theta)}. \tag{4.6}$$

This expression makes F a complex function of argument r only, defined on a bounded ring $r_1 \leq r \leq r_2$, on which it can be generated in Laurent's series. Since the domain of definition is bounded, then the function is a constant [5]:

$$F(Z) = \text{const.}$$

Consequently, p and $\gamma+\theta$ will take constants values and we can write respectively

$$p = D, \quad \gamma + \theta = B, \tag{4.7}$$

such that

$$F(Z) = (\lambda_1 + 2\lambda_2)\left[D - \frac{2(\lambda_1 + \lambda_2)}{\lambda_1 + 2\lambda_2}\right] e^{i(\gamma+\theta)}. \tag{4.8}$$

From (4.5) we have

$$\left(v' + \frac{v(r)}{r}\right)^2 + (v\theta')^2 = D^2$$

or

$$[(rv)']^2 + r^2 v^2 \theta'^2 = r^2 D^2 \tag{4.9}$$

and

$$\cos\gamma = \cos(B - \theta) = \frac{1}{D}\left(v' + \frac{v}{r}\right), \ \ \sin\gamma = \sin(B - \theta) = \frac{1}{D} v\theta'. \tag{4.10}$$

From (4.10), we obtain

$$\frac{(rv)'}{rv} = \theta' \cot(B - \theta).$$

On integration, we obtain

$$rv(r)\sin(B - \theta) = C, \tag{4.11}$$

where C is another constant of integration which, along with others, are found from the boundry conditions. At the first level, these conditions are:

$$v(r_1) = r_1, \ \ v(r_2) = r_2; \ \ \theta(r_1) = \alpha, \ \ \theta(r_2) = 0. \tag{4.12}$$

By these and (4.11) we deduce the values for B and C from the expressions

$$r_2^2 \sin B = C, \ \ r_1^2 \sin(B - \alpha) = C; \ \ \frac{r_2^2}{r_1^2} = \frac{\sin(B - \alpha)}{\sin B}. \tag{4.13}$$

Now, substituting for $rv(r)$ from (4.11) into (4.9) we obtain

$$\frac{\theta'}{\sin^2(B - \theta)} = \frac{rD}{\sin B\, r_1^2}$$

or

$$\frac{d\theta}{\sin^2(B - \theta)} = \frac{D}{\sin B\, r_1^2} r dr. \tag{4.14}$$

We intergrate this to obtain

$$\cot(B-\theta) = \frac{1}{2}\frac{D}{\sin B\, r_1^2}r^2 + A. \tag{4.15}$$

By using the boundary conditions (4.12) we find the constants A, D:

$$D = -2\frac{\sin\alpha}{\sin(B-\alpha)}\frac{r_2^2}{r_2^2 - r_1^2}, \tag{4.16}$$

$$A = \frac{r_2^2}{r_2^2 - r_1^2}\cot(B-\alpha) - \frac{r_1^2}{r_2^2 - r_1^2}\cot B. \tag{4.17}$$

Finally, from (4.15) we obtain

$$\cot(B-\theta) = \frac{r_2^2 - r^2}{r_2^2 - r_1^2}\cot(B-\alpha) - \frac{r_1^2 - r^2}{r_2^2 - r_1^2}\cot B. \tag{4.18}$$

Thus, we obtain the solutions (4.11) and (4.18) to the posed problem.

5. Example: torsion of tube through a shaft

We consider a tube of concentric cylinders, with inner radius $r = r_1$ and outer radius $r = r_2$. Suppose the tube is subjected to torsion via an inserted rigid shaft that is turned through a constant angle, $\theta = \alpha$. The exterior surface is fixed, $\theta = 0$. We are interested in the radial position of fibre elements of the tube in the current configuration (i.e., what has become of a fibre that was radial prior to deformation ?).

Following from the theory in the previous section, we observe that in complex variable formulation, the equilibrium equations at the first level reduce to (3.11). This in turn leads to (3.12) or (4.6). So, on the strength of relations (4.5) and conditions (4.7), the equilibrium equation gave rise to (4.9) and (4.10). However, (4.10) is intergrated to obtain (4.11).

Now, we rewrite (4.9) as

$$\left[\frac{(rv)'}{r}\right]^2 + (v^2\theta') = 2D_0\,, \tag{5.1}$$

where $2D_o = D$ is a constant to be determined. Also, from (4.11) we obtain

$$\sin(B-\theta) = \frac{C}{rv}, \quad \cos(B-\theta) = \sqrt{1 - \left(\frac{C}{rv}\right)^2}. \tag{5.2}$$

Putting this in (5.1) and integrating the result gives

$$\sqrt{(rv)^2 - C^2} = D_0 r^2 + K. \tag{5.3}$$

Thus, we obtain (4.11) and (5.3) as the general solutions of the problem. These can be put in the form

$$v^2 = \left(D_0 r + \frac{K}{r}\right)^2 + \frac{C^2}{r^2} \tag{5.4}$$

and

$$\theta = B - \sin^{-1}\frac{C}{rv}, \tag{5.5}$$

where the constants B, C, D_0 and K are found from the 4 boundary conditions of the problem

$$\theta(r_1) = \alpha,\ \ \theta(r_2) = 0, \quad v(r_1) = r_1,\ \ v(r_2) = r_2. \tag{5.6}$$

The problem is completely resolved when the values of these constants are reflected in the general solution. To this end, we evoke the following non-dimensional parameters

$$\rho = \frac{r}{r_1}, \quad u = \frac{v}{r_1}, \quad a = \frac{r_2}{r_1}, \tag*{(5.6)$'$}$$

where $r_2 > r_1$ and a is a geometric constant. Then, the boundary conditions (5.6) become

$$\theta(1) = \alpha,\ \ \theta(a) = 0, \quad v(1) = 1,\ \ v(a) = a. \tag{5.7}$$

Now, by the first two conditions in (5.7) we deduce the constants B and C from (4.11). In fact, in view of (5.7) we have

$$\sin(B - \alpha) = C, \quad a^2 \sin B = C$$

such that

$$\sin B \cos\alpha - \cos B \sin\alpha = C$$

and

$$\sin B = \frac{C}{a^2}, \quad \cos B = \sqrt{1 - \left(\frac{C}{a^2}\right)^2}.$$

Then

$$B = \sin^{-1}\frac{\sin\alpha}{\gamma}, \quad C = \frac{a^2 \sin\alpha}{\gamma}, \tag{5.8}$$

where $\gamma = \sqrt{a^4 - 2a^2\cos\alpha + 1}$.

Also, the last two conditions in (5.7) enable us to find D_0 and K from (5.4). In fact, from (5.4)

$$D_0 + K = \sqrt{1 - C^2}, \quad a^2 D_0 + K = \sqrt{a^4 - C^2}.$$

Solving these expressions simultaneously, using the value of C given in (5.8), we obtain

$$D_0 = -\frac{1 + a^2}{\gamma}, \quad K = 2\frac{a^2\cos^2\frac{\alpha}{2}}{\gamma}. \tag{5.9}$$

Consequently, from (5.4) and (5.5), on the basis of (5.8) and (5.9), we obtain the final solutions as

$$u(\rho) = \sqrt{\frac{1}{\gamma^2}\left\{\left[\left(2a^2\cos^2\frac{\alpha}{2}\right)\frac{1}{\rho} - (1+a^2)\rho\right]^2 + (a^4\sin^2\alpha)\frac{1}{\rho^2}\right\}} \tag{5.10}$$

and

$$\theta(\rho) = \sin^{-1}\left(\frac{\sin\alpha}{\gamma}\right) - \sin^{-1}\left(\frac{\sin\alpha}{\gamma}\frac{a^2}{\rho u}\right). \tag{5.11}$$

A cursory look at the relations reveals that there is a change in radial magnitude of material fibre as a result of torsion from the reference configuration to the current configuration. This is a nonlinear phenomenon, due to finite deformation approach. It is often not explicitly noticed by the linear theory of the classical elasticity (or the small deformation theory of elasticity) [6].

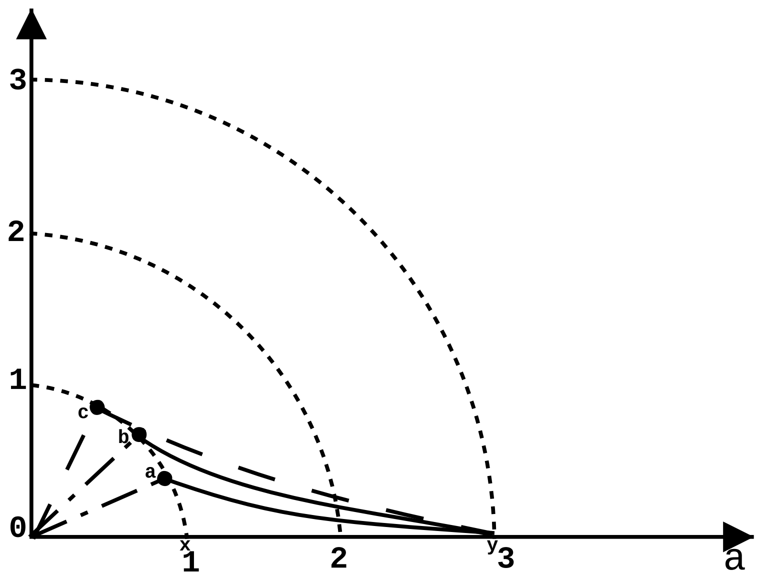

FIGURE 1. The curves ay, by and cy represent the current configurations of a radial fibre which was segment $[x, y]$ in the initial configuration.

In fact, in Figure 1, the curves represent the current configurations of the fibre xy (defined by $\varphi = \text{const}$, in the reference configuration), as the layer is subjected to torsion, by turning an inserted rigid shaft of equal radius as the internal radius of the tube $a = 1$, in angle $\theta = 30^o$, 45^o and 60^o respectively. The tube's thickness is $1 < a < 3$. It is observed that the value of the displacement, $u(\rho) - \rho$, is non-positive. i.e., fibre elements of the tube reduce in radial length as the tube undergoes torsion, and as we move away from the interior layer the angle of displacement of a fibre decreases. This implies that the shaft, indeed, experiences what we call a "hold" (or "grip") from the torsioned tube.

6. Conclusion

By invoking homogenization concept a nonlinear problem of heterogeneous medium has been converted to that of homogeneous but anisotropic problem. With the aid of anisotropic expansion we have been able to distil from a nonlinear anisotropic problem a recurrence system of isotropic problems and consider the problem at the first level.

Relation (4.11) and (4.18) allow us to find the position (radius $r(R)$ and angle $\theta(R)$) of a particle in the current configuration Ω; where, the 4 boundary conditions specified by (4.12) are employed to establish the 4 constants of integration, A, B, C, D. Those two relations thus constitute a solution of the problem. The specific example of a tube subjected to torsion from the interior by a rigid shaft is solved completely. By (5.10) and (5.11), it is seen that the radial fibres of the tube decrease in length. This confirms the phenomenon referred to as "hold effect".

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