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SOME PROPERTIES AND APPLICATION OF THE EXTENDED TYPE I GENERALIZED LOGISTIC DISTRIBUTION

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ABSTRACT

In this paper, we present some results on properties of extended type I generalized logistic distribution of Olapade (2004). The mean, the median, the mode, the 100 *k*-percentage point of the distribution are presented. The study of order statistics is also considered in addition to the estimation of the parameters of the distribution and an application of the distribution to model a social data.

KEYWORDS: Extended type I generalized logistic distribution, mean, median, mode, 100 *k*-percentage point, order statistics, estimation and application.

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INTRODUCTION

The logistic model has been playing very prominent role in the analyses of biological assay and quantal response data. George and Ojo (1980), Ojo (1989), Ojo (1997), Ojo (2002), Olapade (2002) are few of many publications in which logistic model had been studied and applied. However, for the reason of flexibility, generalized models have continued to attract attention. Meenakshi *et al* (1993) applied generalized logistic models for analyzing low-dose response data. Olapade (2004) presented a form of generalized logistic distribution named extended type I generalized logistic distribution with probability density function

$$f_X(x;\lambda,p) = \frac{p\lambda^p e^{-x}}{(\lambda+e^{-x})^{p+1}}, -\infty < X < \infty, \lambda > 0, p > 0 \tag{1}$$

and the corresponding cumulative distribution function

$$F_X(x;\lambda,p) = \frac{\lambda^p}{(\lambda+e^{-x})^p}, -\infty < X < \infty, p > 0, \lambda > 0, \tag{2}$$

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for a continuous random variable X. When $p = \lambda = 1$, we have the ordinary logistic distribution function and when $\lambda = 1$, we have the type I generalized logistic distribution of Balakrishnan and Leung (1988). In this paper, we present some properties of this distribution and applied it to fit a social data.

1. THE MOMENTS OF THE EXTENDED TYPE I GENERALIZED LOGISTIC DISTRIBUTION

The moment generating function of the extended type I generalized logistic distribution is given as

$$M_X(f) = E[e^{tx}] = \int_{-\infty}^{\infty} e^{tx} f_X(x)dx$$

$$M_X(f) = p\lambda^p \int_0^{\infty} \frac{y^{-t}}{(\lambda + y)^{p+1}} dy = \frac{p}{\lambda} \int_0^{\infty} \frac{y^{-t}}{(1 + y/\lambda)^{p+1}} dy.$$

Now, let $y/\lambda = z, y = \lambda z$, and $dy = \lambda dz$.. Hence

$$M_X(f) = p\lambda^{-t} \int_0^{\infty} \frac{z^{-t}}{(1 + z)^{p+1}} dz = \frac{\lambda^{-t}\Gamma(p + t)\Gamma(1 - t)}{\Gamma(p)} .t < 1, \tag{1.11}$$

where $B(p,q)$ is a complete beta function and it is related to the complete gamma function with the relation $B(p,q) = \Gamma(p)\Gamma(q)/\Gamma(p+q)$. Similarly, the characteristic function of the extended type I generalized logistic distribution is obtained as

$$\Phi_X(t) = \frac{\lambda^{-it}\Gamma(p+it)\Gamma(1-it)}{\Gamma(p)} \tag{1.2}$$

The cumulant generating function is given as

$$\ln \varphi(t) = -t \ln \lambda + \ln \Gamma(p+t) + \ln \Gamma(1-t) - \ln \Gamma(p). \tag{1.3}$$

The r^{th} cumulant is obtained as

$$\kappa_r(X) = \frac{d^r}{dt^r}[-t \ln \lambda]_{t=0} + \frac{d^r}{dt^r}[\ln \Gamma(p+t) + \ln \Gamma(1-t)]_{t=0}. \tag{1.4}$$

The 2^{nd} or the 3^{rd} term of the right hand side of equation (1.4) is called di-gamma function. The series expansion of this function as in Copson (1962) is given as

$$\psi^{r-1}(x)=(r-1)![(-1)^r\sum_{j=0}^{\infty}(j+x)^{-r}]_{r\geq 2}$$

and

$$\psi(x)=\sum_{j=0}^{\infty}(j+x)^{-1},\ r=1.$$

Therefore

$$\kappa_r(X)=(r-1)!(-1)^r[\sum_{j=0}^{\infty}(j+p)^{-r}+(-1)^r\sum_{j=0}^{\infty}(j+1)^{-r}]_{r\geq 2} \tag{1.5}$$

and

$$\kappa_r(X)=-\ln\lambda+\sum_{j=0}^{\infty}(j+1)^{-1}-\sum_{j=0}^{\infty}(j+p)^{-1},\ r=1. \tag{1.6}$$

When $r=1$, we obtained the mean μ_1 of the extended type I generalized logistic distribution as

$$\mu_1=\kappa_1(X)=-\ln\lambda+\sum_{j=0}^{\infty}(j+1)^{-1}-\sum_{j=0}^{\infty}(j+p)^{-1}$$

$$=-\ln\lambda+\sum_{j=1}^{p-1}j^{-1}. \tag{1.7}$$

The second cumulant of the distribution is obtained when $r=2$ as

$$\kappa_2(X)=\sum_{j=0}^{\infty}(j+p)^{-2}+\sum_{j=0}^{\infty}(j+1)^{-2}$$

$$=2\sum_{j=p}^{\infty}j^{-2}+\sum_{j=1}^{p-1}j^{-2}. \tag{1.8}$$

Therefore, the variance of the random variable X can be obtained from $\kappa_2(X)-\{\kappa_1(X)\}^2$ as

$$\mu_2=\sigma_X^2=2\sum_{j=p}^{\infty} j^{-2}+\sum_{j=1}^{p-1} j^{-2}-(\sum_{j=1}^{p-1} j^{-1}-\ln\lambda)^2. \tag{1.9}$$

The third cumulant of the distribution when $r=3$ in equation (1.5) is

$$\kappa_3(X)=-2\{\sum_{j=0}^{\infty}(j+p)^{-3}-\sum_{j=0}^{\infty}(j+1)^{-3}\}=2\sum_{j=1}^{p-1} j^{-3}. \tag{1.10}$$

The fourth cumulant of the random variable X that has an extended type I generalized logistic distribution is obtained when $r=4$ as

$$\kappa_4(X)=6\{\sum_{j=0}^{\infty}(j+p)^{-4}+\sum_{j=0}^{\infty}(j+1)^{-4}\}=12\sum_{j=p}^{\infty} j^{-4}+6\sum_{j=1}^{p-1} j^{-4}. \tag{1.11}$$

So, the third moment μ_3 of the distribution could be obtained from

$$\mu_3=\kappa_3(X)-3\kappa_2(X)\kappa_1(X)+2\{\kappa_1(X)\}^3 \tag{1.12}$$

while the fourth moment could be obtained from

$$\mu_4=\kappa_4(X)-4\kappa_3(X)\kappa_1(X)+6\kappa_2\{\kappa_1(X)\}^2-3\{\kappa_1(X)\}^4. \tag{1.13}$$

Hence, the coefficient of skewness and that of kurtosis of the random variable X with extended type I generalized logistic distribution are $\beta_1(X)=\mu_3^2/\mu_2^3$ and $\beta_2(X)=\mu_4/\mu_2^2$ respectively.

2. THE MEDIAN OF THE EXTENDED TYPE I GENERALIZED LOGISTIC DISTRIBUTION

The median of a probability density function $f(x)$ is a point x_m on the real line which satisfies the equation

$$\int_{-\infty}^{x_m} f(x)\,dx=1/2.$$

This implies that $F(x_m)=1/2$. For the extended type I generalized logistic distribution with probability distribution function

$$F_X(x;\lambda,p)=\frac{\lambda^p}{(\lambda+e^{-x})^p}.$$

Equating the distribution function to $1/2$ implies

$$\frac{\lambda^p}{(\lambda+e^{-x})^p}=1/2 \Rightarrow \frac{\lambda}{\lambda+e^{-x}}=\frac{1}{\sqrt[p]{2}}$$

$$\Rightarrow \lambda+e^{-x}=\lambda\sqrt[p]{2}$$

$$\Rightarrow x_{median}=-\ln\lambda-\ln(\sqrt[p]{2}-1). \tag{2.1}$$

This reduces to zero when $\lambda=1$ and $p=1$ as the median of the standard logistic distribution.

3. THE 100 *k*-PERCENTAGE POINT OF THE EXTENDED TYPE I GENERALIZED LOGISTIC DISTRIBUTION

Consider the extended type I generalized logistic distribution, the $100k$ -percentage point is obtained by equating the probability distribution function to k, where $0 \le k \le 1$. That is

$$F(x_{(k)})=k \Rightarrow \frac{\lambda^p}{(\lambda+e^{-x})^p}=k$$

$$\Rightarrow \frac{\lambda}{\lambda+e^{-x}}=\sqrt[p]{k}.$$

Solving for $x_{(k)}$ gives

$$x_{(k)}=-\ln\left[\frac{\lambda(1-\sqrt[p]{k})}{\sqrt[p]{k}}\right]. \tag{3.1}$$

This gives the value of the point $x_{(k)}$ on the real line that produce a percentage k of the distribution. We can easily test this by checking the value of $x_{(k)}$ when $k=0.5$ which corresponds to the median when $p=\lambda=1$. This gives the value of the median for a standard logistic distribution which is already known to be $\ln 1=0$.

4. THE MODE OF THE EXTENDED TYPE I GENERALIZED LOGISTIC DISTRIBUTION

The mode of a probability density function is obtained by equating the derivative of the density function to zero and solve for the variable. Therefore, for the extended type I generalized logistic distribution

$$f_X(x;\lambda,p)=\frac{p\lambda^p e^{-x}}{(\lambda+e^{-x})^{p+1}}.$$

$$f'_X(x;\lambda,p)=p\lambda^p[\frac{-e^{-x}(\lambda+e^{-x})+(p+1)e^{-2x}}{(\lambda+e^{-x})^{p+2}}]. \tag{4.1}$$

By equating the derivative to zero, we have

$$p\lambda^p[\frac{pe^{-2x}-\lambda e^{-x}}{(\lambda+e^{-x})^{p+2}}]=0\Rightarrow x=\ln p-\ln\lambda. \tag{4.2}$$

This gives the mode of the distribution as confirmed when we put $p=\lambda=1$ which gives the mode of standard logistic distribution.

5. ORDER STATISTICS FROM THE EXTENDED TYPE I GENERALIZED LOGISTIC DISTRIBUTION

Let $X_1, X_2, \ldots, X_n$ be n independently continuous random variables from the extended type I generalized logistic distribution and let $X_{1:n} \le X_{2:n} \le \ldots \le X_{n:n}$ be the corresponding order statistics. Let $F_{X_{r:n}}(x)$, $r=1,2,\ldots,n$ be the cumulative distribution function of the r^{th} order statistics $X_{r:n}$ and $f_{X_{r:n}}(x)$ denotes its probability density function. David (1970) gives the probability density function of $X_{r:n}$ as

$$f_{X_{r:n}}(x)=\frac{1}{B(r,n-r+1)}P^{r-1}(x)[1-P(x)]^{n-r}p(x). \tag{5.1}$$

For the extended type I generalized logistic distribution with probability density function and cumulative distribution function given in equations (1) and (2) respectively, then

$$f_{X_{r:n}}(x)=\frac{1}{B(r,n-r+1)}[\frac{\lambda^p}{(\lambda+e^{-x})^p}]^{r-1}[1-\frac{\lambda^p}{(\lambda+e^{-x})^p}]^{n-r}\frac{p\lambda^p e^{-x}}{(\lambda+e^{-x})^{p+1}}$$

$$=\frac{p\lambda^p}{B(r,n-r+1)}\frac{e^{-x}}{(\lambda+e^{-x})^{p+1}}(\frac{\lambda}{\lambda+e^{-x}})^{pr-p}(\frac{(\lambda+e^{-x})^p-\lambda^p}{(\lambda+e^{-x})^p})^{n-r}$$

$$=\frac{p\lambda^{pr}e^{-x}((\lambda+e^{-x})^p-\lambda^p)^{n-r}}{B(r,n-r+1)(\lambda+e^{-x})^{pn+1}}. \tag{5.2}$$

5.1. THE MAXIMUM ORDER STATISTICS

Consider the probability density of the r^{th} order statistics from the extended type I generalized logistic distribution in equation (5.2). Let $r = n$, then the probability density function of the maximum order statistic is

$$f_{X_{n:n}}(x) = \frac{np\lambda^{np} e^{-x}}{(\lambda + e^{-x})^{pn+1}}. \tag{5.3}$$

This is another extended type I generalized logistic distribution with parameters (np, λ). This distribution shares all the properties of the extended type I generalized logistic distribution with np replacing p.

5.2. THE MINIMUM ORDER STATISTICS

Consider the probability density of the r^{th} order statistics from the extended type I generalized logistic distribution in equation (5.2). Let $r = 1$, then the probability density function of the minimum order statistic is obtained as

$$f_{X_{1:n}}(x) = \frac{np\lambda^{p} e^{-x}((\lambda + e^{-x})^{p} - \lambda^{p})^{n-1}}{(\lambda + e^{-x})^{pn+1}}$$

$$= \frac{np\lambda^{p} e^{-x}[\sum_{k=0}^{n-1}(-1)^{k}\binom{n-1}{k}(\lambda + e^{-x})^{pk}\lambda^{p(n-k-1)}]}{(\lambda + e^{-x})^{pn+1}}$$

$$= np\sum_{k=0}^{n-1}\sum_{j=0}^{pk-pn-1}(-1)^{k}\binom{n-1}{k}\binom{pk-pm-1}{j}\lambda^{-j-1}e^{-(j+1)x}. \tag{5.4}$$

The q^{th} moment of $X_{1:n}$ is obtained as

$$E[X_{1:n}^{q}] = np\sum_{k=0}^{n-1}\sum_{j=0}^{pk-pn-1}(-1)^{k}\binom{n-1}{k}\binom{pk-pm-1}{j}\lambda^{-j-1}\int_{-\infty}^{\infty}x^{q}e^{-(j+1)x}dx. \tag{5.5}$$

The equation (5.5) above can be used to obtain the moments of the minimum observations from the extended type I generalized logistic distribution.

6. ESTIMATION OF THE PARAMETERS OF THE EXTENDED TYPE I GENERALIZED LOGISTIC DISTRIBUTION

Given a sample $X_1, X_2, \ldots, X_n$ of size n from an extended type I generalized logistic distribution with probability density function

$$f_X(x;\mu,\sigma,\lambda,p)=\frac{p\lambda^p}{\sigma}\frac{e^{-(\frac{x_i-\mu}{\sigma})}}{[\lambda+e^{-(\frac{x_i-\mu}{\sigma})}]^{(p+1)}},-\infty<X<\infty,-\infty<\mu<\infty,\sigma>0,p>0,\lambda>0, \quad (6.1)$$

where μ is the location parameter, σ is the scale parameter, p is the shape parameter and λ is a shift parameter. The likelihood function of the extended type I generalized logistic distribution is given as

$$L(X;\mu,\sigma,\lambda,p)=\frac{p^n\lambda^{np}}{\sigma^n}\frac{e^{-\sum_{i=1}^{n}(\frac{x_i-\mu}{\sigma})}}{\prod_{i=1}^{n}[\lambda+e^{-(\frac{x_i-\mu}{\sigma})}]^{(p+1)}} \quad (6.2)$$

Taking the natural logarithm of $L(X;\mu,\sigma,\lambda,p)$, we have

$$\ln L(X;\mu,\sigma,\lambda,p)=n\ln p+np\ln\lambda-n\ln\sigma-\sum_{i=1}^{n}(\frac{x_i-\mu}{\sigma})-(p+1)\sum_{i=1}^{n}\ln[\lambda+e^{-(\frac{x_i-\mu}{\sigma})}]. \quad (6.3)$$

To obtain the estimates of the parameters that maximize the likelihood function, we differentiate the logarithm of the likelihood function partially with respect to each of the parameters and equate the derivatives to zero and solve for the parameters. Hence by differentiation

$$\frac{\partial \ln L(X;\mu,\sigma,\lambda,p)}{\partial\mu}=\frac{n}{\sigma}-\frac{(p+1)}{\sigma}\sum_{i=1}^{n}\frac{e^{-(\frac{x_i-\mu}{\sigma})}}{[\lambda+e^{-(\frac{x_i-\mu}{\sigma})}]}. \quad (6.4a)$$

$$\frac{\partial \ln L(X;\mu(X;\mu p)}{\partial\sigma}=\frac{n}{\sigma}+\frac{\sum_{i=1}^{n}(x_i-\mu)}{\sigma^2}-\frac{(p+1)}{\sigma^2}\sum_{i=1}^{n}\frac{(x_i-\mu)e^{-(\frac{x_i-\mu}{\sigma})}}{[\lambda+e^{-(\frac{x_i-\mu}{\sigma})}]} \quad (6.4b)$$

$$\frac{\partial \ln L(X;\mu,\sigma,\lambda,p)}{\partial \lambda} = \frac{np}{\lambda} - (p+1)\sum_{i=1}^{n} \frac{1}{[\lambda + e^{-(\frac{x_i-\mu}{\sigma})}]}. \tag{6.4c}$$

$$\frac{\partial \ln L(X;\mu,\sigma,\lambda,p)}{\partial p} = \frac{n}{p} - n\ln\lambda - \sum_{i=1}^{n} \ln[\lambda + e^{-(\frac{x_i-\mu}{\sigma})}]. \tag{6.4d}$$

Since the equations (6.4a,b,c) above are nonlinear in the parameters, we used numerical iterative method with the aid of python optimization computer programme to estimate the parameters from a given sample.

7. APPLICATION OF THE EXTENDED TYPE I GENERALIZED LOGISTIC MODEL

A data on the age of marriage in years before divorce was obtained from a Grade C Customary Court. The distribution of a total of 1553 marriages that were divorced in a particular period of five years are shown in table below. The histogram of this data shows that it is positively skewed. Hence we decided to fit two positively skewed distributions namely the extended type I generalized logistic and the type I generalized logistic distribution of Balakrishnan and Leung (1988) to the data.

The estimates of the parameters of the extended type I generalized logistic distribution as obtained from the python computer programme are $(\mu = 17.5, \sigma = 6.3, \lambda = 3.9, p = 2.5)$ while that of the type I generalized logistic of Balakrishnan and Leung are $(\mu = 16.9, \sigma = 6.15, p = 0.91)$. The outcomes of the analysis are shown in the table below.

The adequacy of the model is tested using the method of chi-square test of adequacy given as

$$\chi^2_{calculated} = \sum_{i=1}^{n} \frac{(O_i - E_i)^2}{E_i} \tag{7.1}$$

where O_i = observed frequency from the data, E_i =expected frequency of the model. The statistic $\chi^2_{calculated}$ is to be compared with $\chi^2_{(n-k,\alpha)}$ from the chi-square table where n is the number of classes, k is the number of parameters estimated from the data and α is the confidence level. If $\chi^2_{calculated} \le \chi^2_{table}$, the model is adequate. On the other hand, if two models are compared, the model with smaller $\chi^2_{calculated}$ value is to be preferred. Generally, the closer the $\chi^2_{calculated}$ value is to zero, the better.

From the table below, $\chi^2_{calculated}$ for the extended type I generalized logistic model is 10.2873 while that of type I generalized logistic of Balakrishnan and Leung is 43.34. From the table of chi-square, $\chi^2_{(5,0.05)} = 11.0705$ and $\chi^2_{(7,0.05)} = 14.0671$. While extended type I generalized logistic model is adequate for this data, the type I generalized logistic model is inadequate.

Length of marriage in years	Class mid-point x	Class frequency obsereved	Ext. type I estimated frequency	Type I estimated frequency	χ^2 Ext. Type I gen. log	χ^2 Type I gen. log
≤5	3	127	96.4	122.8	9.713	0.014
6-10	8	247	241.0	205.6	0.149	8.34
11-15	13	361	362.4	283.3	0.005	21.31
16-20	18	347	342.9	287.2	0.049	8.34
21-25	23	240	231.9	232.3	0.283	0.26
26-30	28	125	128.3	142.4	0.085	2.13
31-35	33	64	63.8	74.6	0.001	1.51
36-40	38	30	30.2	35.8	0.0013	0.94
41-45	43	14	13.9	16.5	0.001	0.38
Total		1553			10.2873	43.34

Distribution of age of marriage in years before divorce in a customary court

CONCLUSIONS

We have established some properties of the extended type I generalized logistic distribution. The moments μ_1, μ_2, μ_3 and μ_4 have been theoretically obtained and the mean μ, variance σ^2, skewness β_1 and kurtosis β_2 have been established. The median, the $100k$-percentage point and the mode of the distribution were presented. The distributions of the r^{th} order statistics $X_{r:n}$, the maximum order statistics $X_{n:n}$ and the minimum order statistics $X_{1:n}$ of the distribution are also established. Estimation of the parameters of the distribution and an application in modelling a social data conclude the paper. Further generalization of the extended type I generalized logistic distribution which contains four parameters to a five-parameter extended type I generalized logistic distribution is currently under study.

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