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ON THE RADIUS OF STARLIKENESS AND CONVEXITY OF CERTAIN SUBCLASS OF ANALYTIC FUNCTIONS

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Abstract

Let $A(n)$ denote the subclass of A consisting of all functions of the form: $f(z) = z - \sum_{k=n+1}^{\infty} a_k z^k$ $(a_k \geq 0;\, n \in N)$. In this paper, we investigate the radius of starlikeness and convexity of the function $f(z) = (z - w) - \sum_{k=n+1}^{\infty} e^{i(k-1)\theta} a_k (z - w)^k$.

1. Introduction

Let $H(u)$ be the class of functions analytic in the unit disk $U = \{z \in C : |z| < 1\}$ and of the form:

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$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1}$$

$$A = \{f \in H(u) : f(0) = f'(0) - 1 = 0\},$$

$$S = \{f \in A : f \text{ is univalent in } U\}$$

$$S^* = \left\{f \in S : \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0,\ z \in U\right\},$$

$$S^c = \left\{f \in S : \operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0,\ z \in U\right\}.$$

Kanas and Ronning [3] introduced the following classes of functions:

$$A(w) = \{f \in H(u) : f(w) = f'(w) - 1 = 0\},$$

$$S(w) = \{f \in A(w) : f \text{ is univalent in } U\},$$

$$S^*(w) = \left\{f \in S : \operatorname{Re}\left(\frac{(z-w)f'(z)}{f(z)}\right) > 0,\ z \in U\right\},$$

$$S^c(w) = \left\{f \in S : \operatorname{Re}\left(1 + \frac{(z-w)f''(z)}{f'(z)}\right) > 0,\ z \in U\right\},$$

where $f \in A$ is of the form

$$f(z) = (z - w) + \sum_{k=2}^{\infty} a_k (z - w)^k. \tag{2}$$

The classes $A(w)$, $S(w)$, S^* and $S^c(w)$ are, respectively, analytic, univalent, starlike and convex functions with respect to a fixed point $w \in U$.

Let f be as in (1). Then f is said to be α-*starlike* if $\operatorname{Re}\left(\frac{(z-w)f'(z)}{f(z)}\right) > \alpha$.

Also, let $A(n)$ denote the subclass of A consisting of all functions of the form: $f(z) = z - \sum_{k=n+1}^{\infty} a_k z^k$ $(a_k \geq 0;\ n \in N)$. $A(n)$ is said to be the *class* of analytic functions with negative coefficients. Acu and Owa [1] introduced a subclass $A(n, \theta)$

of $A(n)$ consisting of all functions of the form: $f(z) = z - \sum_{k=n+1}^{\infty} e^{i(k}$

$(a_k \geq 0;\ n \in N)$.

2. Preliminaries

Lemma 1 [2]. *A function* $f(z)$ *in* $A(n, \theta)$ *is* α-*starlike if and only if* $\sum_{k=n+1}^{\infty} (k - \alpha) a_k \leq 1 - \alpha$.

3. The Main Results

We now give the proof of the following results:

Let $A(n, \theta)$ be the class of functions of the form:

$$f(z) = (z - w) - \sum_{k=n+1}^{\infty} e^{i(k-1)\theta} a_k (z - w)^k \quad (a_k \geq 0;\ n \in N). \tag{3}$$

Theorem 1. *A function* $f(z)$ *in* $A(n, w, \theta)$ *is* α-*starlike if and only if* $\sum_{k=n+1}^{\infty} (k - \alpha) a_k \leq 1 - \alpha$ *holds.*

Proof. We first show the sufficient condition. Let $f(z)$ be as in (3). Then

$$\left|\frac{(z-w)f'(z)}{f(z)} - 1\right|$$

$$= \left|\frac{(z-w) - \sum_{k=n+1}^{\infty} e^{i(k-1)\theta} k a_k (z-w)^k - \left[(z-w) - \sum_{k=n+1}^{\infty} e^{i(k-1)\theta} a_k (z-w)^k\right]}{(z-w) - \sum_{k=n+1}^{\infty} e^{i(k-1)\theta} a_k (z-w)^k}\right|$$

$$\leq \frac{\sum_{k=n+1}^{\infty} (k-1) a_k |z-w|^{k-1}}{1 - \sum_{k=n+1}^{\infty} a_k |z-w|^{k-1}}$$

$$\leq \frac{\sum_{k=n+1}^{\infty} (k-1) a_k}{1 - \sum_{k=n+1}^{\infty} a_k} \quad (\text{since } z,\ w \in U)$$

by hypothesis we have

$$\left|\frac{(z-w)f'(z)}{f(z)}-1\right| \le \frac{\sum_{k=n+1}^{\infty}(k-1)a_k}{1-\sum_{k=n+1}^{\infty}a_k} \le \frac{1-\alpha}{1-\sum_{k=n+1}^{\infty}a_k}$$

$$\le 1-\alpha.$$

Thus, $f(z)$ is α-starlike

Now, the necessary condition:

Let $f \in S_\alpha^*(w)$. Then we have

$$\mathrm{Re}\left\{\frac{(z-w)f'(z)}{f(z)}\right\} = \mathrm{Re}\left\{\frac{(z-w)-\sum_{k=n+1}^{\infty}ka_k(z-w)^k}{(z-w)-\sum_{k=n+1}^{\infty}a_k(z-w)^k}\right\} > \alpha,$$

$z,\ w \in U;\ z-w = re^{i\theta}, \quad 0 \le r < 1,$

$$\mathrm{Re}\left\{\frac{(z-w)f'(z)}{f(z)}\right\} = \frac{1-\sum_{k=n+1}^{\infty}ka_kr^{k-1}}{1-\sum_{k=n+1}^{\infty}a_kr^{k-1}} > \alpha$$

which implies $\sum_{k=n+1}^{\infty}(k-\alpha)a_k < 1-\alpha$ as $r \to 1$ and $1-\sum_{k=n+1}^{\infty}a_kr^{k-1} > 0$. Thus, the result holds.

Theorem 2. *Let $f(z)$ be as in (3). Then $f(z)$ is α-convex if* $\sum_{k=n+1}^{\infty}k^2a_k < \alpha$, $0 \le \alpha < 1$.

Proof. For f as in (3), we have

$$\left|1+\frac{zf''(z)}{f'(z)}-1\right| = \left|\frac{zf''(z)}{f'(z)}\right|$$

$$\le \frac{1-\sum_{k=n+1}^{\infty}k^2a_k|z-w|^{k-1}}{1-\sum_{k=n+1}^{\infty}ka_k|z-w|^{k-1}}$$

$$\le \frac{1-\sum_{k=n+1}^{\infty}k^2a_k}{1-\sum_{k=n+1}^{\infty}ka_k}$$

$$\le \frac{1-\alpha}{1-\sum_{k=n+1}^{\infty}ka_k} \quad \text{(by hypothesis)}$$

$$\le 1-\alpha.$$

Thus, $\mathrm{Re}\left\{1+\frac{(z-w)f''(z)}{f'(z)}\right\} > \alpha.$

Hence, $f(z)$ is α-convex.

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